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# Plane and Solid
# Geometry

# Plane and Solid Geometry

## *Suggestive Method*


By

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Atkinson, Mentzer & Grover

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# THE PREFACE.

THIS present book is a revision of the Van Velzer and Shutts Plane and Solid Geometry, Suggestive Method, published by Tracy, Gibbs & Company. The method of the book was used by the author for several years from mimeograph reprints. This text was then revised and incorporated into text-book form. After being again thoroughly tested in many of the best schools in the country, the work has been again revised in an attempt to make it more suggestive to the teacher and more helpful to the student.

In putting the work into its present form the scientific classification of the subject-matter has been departed from when it was thought that by so doing the work could be better graded to the ability of the average pupil. For this reason the subject of the triangle has been introduced before the relation of lines and angles has been fully discussed.

The treatment of the theory of measurement has been modified to make it more easily understood. A treatment of the application of proportion has been suggested that will make the pupil more independent in his work, and at the same time it has not increased the difficulty of the subject.

The book, as now arranged, is sufficient for all college entrance requirements, yet it can be completed, including all the exercises, by high-school pupils in one school year. For those schools that devote a year and a half to the subject additional work has been placed in the Appendix, to which references are given in the text, so that the various propositions and exercises can be taken up in logical order. A fuller treatment of the theory of limits, which many teachers desire, is also given in the Appendix.

A departure from ordinary methods will be noticed in the treatment of proportion. It has not been thought wise to follow the usual method of limiting the subject to proportions whose terms are pure numbers, nor yet to follow the Euclidian method common in England, which admits of proportions whose terms are concrete magnitudes, but which is so difficult that it can be understood by only the best students. The method in the text will be found to admit of proportions whose terms are concrete magnitudes, yet it will do no violence to the fundamental ideas of Arithmetic regarding operations upon concrete magnitudes. It is believed that the subject of limits is treated in so simple a manner that beginners can grasp it.

An edition, consisting of the theorems, the diagrams, the things given, and the things to prove, *without* suggestions for demonstration, will accompany the book for class-room use. It is hoped that this will be found as valuable in Geometry teaching as "text editions" have been in teaching the classics. This "Class-Room Edition" will save the time of the recitation usually consumed by the pupils in drawing the figures upon the blackboard. The complete edition of the book can be banished from the recitation and the temptation to get assistance from the text eliminated. The blackboards can thus be reserved for original demonstrations of exercises and for suggestive work by the teacher. The Class-Room Edition will lengthen the daily recitation, and make it possible for more work to be done in a year.

The author wishes to thank those who have given the previous edition of the book so kind a reception. It is hoped that the present book will more fully meet the needs of all teachers who wish their pupils to make the largest possible growth in independent thinking in Geometry.

Acknowledgment should be made of the scholarly criticism of Dr. C. A. Van Velzer, *Head of the Department of Mathematics in the University of Wisconsin,* for his helpful services in preparing for publication the manuscript of the first edition of the book.

Thanks are also due for valuable suggestions in reading the proof to Mr. G. E. Bunsa, *Superintendent of Schools, Columbus, Wisconsin;* Mr. R. L. Sandwick, *Principal of the Deerfield Township High School, Highland Park, Illinois;* Miss Genevieve Decker, *Teacher of Mathematics in the High School, Janesville, Wisconsin;* Miss Maud Averill, *Teacher of Mathematics in the High School, Whitewater, Wisconsin,* and to Mr. Frank P. Dodge, *Instructor in Mathematics in the Roxbury Latin School, Roxbury, Massachusetts.*

G. C. S.

*Whitewater Wis., August 25, 1904.*

# SUGGESTIONS TO TEACHERS.

GEOMETRY is essentially a disciplinary study. The value derived from its study is in proportion to the amount of independent thought expended by the pupil. A text-book in Geometry is in the nature of a "key" to the extent to which the demonstrations are written out for the pupil. That part of the work which a pupil can do for himself should not be done for him. The teacher and text-book should furnish the pupil with data and stimulate thought rather than give him a set form of words which he may repeat verbatim, with or without, the ideas which these words should express.

In this Geometry suggestions arranged in logical order take the place of detailed demonstration. These suggestions are intended to stimulate and direct the thought of the pupil so that he may largely work out his own demonstrations.

Model demonstrations are given of a few propositions to show the student the *form* in which they should be presented. The answers to the suggestions, logically arranged, constitute the demonstration. The suggestions should be studied in the order given, for each suggestion usually depends upon the preceding one. The answer to a suggestion should consist of a statement of the relations asked for, together with the authority in full for such statement.

To permit the pupil to ignore the authority is to encourage carelessness, slovenliness, and inaccuracy in demonstration. A common error is to apply authority that does not exactly fit the conditions under consideration. The pupil must understand that the authority should, without exception, be a definition, an axiom, or a previously proved proposition. "It seems so," or, "it looks reasonable," or any expression of judgment will not do. The pupil should be encouraged to search out his own authority, even when the authority is quoted for him in the suggestions, and to use the reference simply for verification. A pride in independent work is a most important factor in securing satisfactory results.

In the preparation of the lesson the pupil should write out his demonstration, noting carefully the form of the "models." This will insure correct form and avoid haziness of thought. During the first few weeks this written work, as well as tests taken in the recitation, should be read by the teacher and returned to the pupil for correction.

The exercises, or at least a part of them, should be demonstrated daily along with the propositions as they occur, and not be studied all together at the end of a chapter.

The best results will be obtained by starting slowly, reviewing frequently, and passing over nothing that is not clearly understood. Since each demonstration involves previous propositions and definitions, facility in demonstration can best be secured by committing to memory each theorem, definition and axiom; for that authority cannot be readily recognized and applied which is imperfectly remembered. The demonstrations should not be committed to memory.

The subject of proportion is probably the most difficult part of Geometry. Clearness of thought in the applications of proportion can be obtained only by careful illustration and rigid demonstration in the theory. To teach the theory of proportion by means of numbers, and then to apply the principles developed to geometric magnitudes and numbers indiscriminately without consideration of the limitations of the various statements, is not scientific. Note 262, page 137, should receive careful attention. In deriving the form $A = m\,B$ from $\frac{A}{B} = m$ the tendency is to claim the multiplication of both members of the equation by B. This is correct if B is a number, but the process is unthinkable if B is a geometric magnitude. $\frac{A}{B}$ means that A is divided or measured by the unit B, hence to say that A contains B, m times, is simply another way of saying that A is m times the unit B, or m B. 12 contains 4 three times ($\frac{12}{4} = 3$) is another form of expression for 12 is equal to 3 fours ($12 = 3 \times 4$). The expression $\frac{1 \text{ foot}}{1 \text{ inch}} = 12$, means the same as the expression 1 foot is equal to 12 inches. In this connection see § 270.

# THE CONTENTS.

## PLANE GEOMETRY.



## SOLID GEOMETRY.

# PLANE GEOMETRY.

## CHAPTER I.

### RECTILINEAR FIGURES.

#### Definitions.

**1.** The block represented in the accompanying figure occupies a limited portion of space. If we imagine the block to be removed, its form or shape can still be retained in the mind. This is true of any object or body.

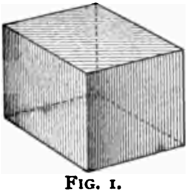


FIG. 1.

The space conceived to be occupied by an object or body as distinguished from the substance of which it is made, is a **geometrical solid.** The matter or substance of which a body or object is composed is a physical solid. Hence a geometrical solid is the shape or form of a physical solid, or some form or figure conceived by the mind.

A **geometrical solid** is a limited portion of space, and has length, breadth, and thickness.

The term solid will be used hereafter to signify a geometrical solid.

**2.** When space is divided into distinct portions or geometrical solids, the boundaries of these portions or solids are **surfaces.** Distinct portions of the bounding surface are **faces.**

**Surface** has length and breadth, but no thickness.

**3.** When a surface is divided into distinct portions, the boundaries of these portions are **lines.** In the solid, represented in Fig. 1, the edges, or boundaries of the faces, are lines. These lines, being the intersection of faces which have no thickness, can themselves have neither breadth nor thickness.

A **line** has length, but neither breadth nor thickness.

**4.** When a line is divided into distinct portions, the limits of these portions are **points.** In the solid, represented in Fig. 1, the corners, or limits of the edges, are points. These points, being the intersections of lines which have neither breadth nor thickness, can themselves have neither length, breadth, nor thickness.

A **point** has position, but neither length, breadth, nor thickness.

**5.** A surface can be conceived of apart from a solid, a line apart from a surface, and a point apart from a line. If a point is conceived to move, the path in which it moves is a line. Hence a line is the path, or **locus,** of a moving point.

A line can be thought of as generated by a **point in motion;** surface can be thought of as generated by a **line in motion;** a solid, as generated by a **surface in motion.**

**6.** A **geometrical figure** is a combination of points, lines, surfaces, or solids.

Geometrical figures are ideal, that is, they are mental conceptions, but they can be represented to the eye only by material substances. For instance, a line can be represented by a mark made by a pencil or crayon; a solid can be represented by a drawing, by a block of wood, or by some other material of any given shape.

To avoid multiplying words, the material representation of geometrical figures will be generally referred to as standing for the mental conceptions themselves. The pupil will be able to tell by the context whether the word "figure" refers to a geometrical figure or to the material representation of a figure.

7. A **straight line** is a line such that any part of it, however placed, lies wholly in any other part if its extremities lie in that part. Let *O B*, which is any part of *A B*, be placed upon some other part in any way, except

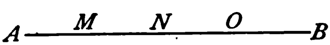



FIG. 2.

that *O* and *B* shall lie upon that part, for instance with *O* at *M* and *B* at *N*. If *O B* exactly coincides with *M N*, *A B* is a straight line. Illustrate by lines represented by wood, paper, or other material.

A line is **read** by naming letters placed at its extremities, as line *A B* in Fig. 3; or by naming a single letter placed upon it, as line *O* in Fig. 9.


FIG 3.

8. A **broken line** is a line made up of a succession of different straight lines, as *A B C D E*, in Fig. 4.

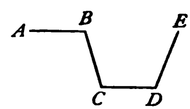



FIG. 4.

9. A **curved line**, or a **curve**, is a line no portion of which is straight, as *C D*, in Fig. 5.


FIG. 5.

10. A **plane surface**, or a **plane**, is a surface such that if any two of its points be joined by a straight line the line lies wholly in the plane surface. If a carpenter wishes to determine whether or not the surface of a board

is a plane, he tries to place a straight edge so that at least two of its points touch the surface. If his straight edge lies continuously in the surface, it is a plane surface.

11. A **plane figure** is a figure that lies wholly in the same plane.

12. A plane figure, in which the lines are all straight lines, is a **rectilinear figure.**

13. **Magnitudes** are figures considered only with reference to extent.

14. **Geometry** is the science that treats of points, lines, surfaces, and solids, and is concerned with the construction and measurement of geometrical figures.

15. **Plane geometry** treats of plane figures.

16. **Solid geometry** treats of figures which are not wholly in the same plane.

## Angles.

17. When two straight lines meet or intersect, they contain, or make with each other, **an angle.**

The two lines are the **sides** or **arms** of the angle, and the point of meeting is its **vertex.**

(*a*) An angle can be **read** by naming the letter at the vertex of the angle between the letters upon the sides of the angle, as angle $A\,B\,C$ or angle $C\,B\,A$ in Fig. 6. When there is only one angle at a given vertex, it is sufficient to read the letter at the vertex, as angle $B$. When two or more angles have a common vertex, letters or figures are frequently placed near the vertex between the sides of the angles to designate the angles, as $m$ and $n$ in Fig 7. For example,

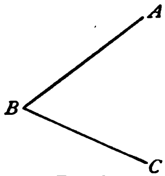



FIG. 6.

we say angle *m* instead of angle *C B A*, and angle *n* instead of angle *C B D*.

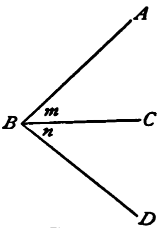


FIG. 7.

(*b*) When a line, coincident with one side of an angle, revolves about the vertex remaining always in the same plane until it arrives at the position of the other side of the angle, the line **turns through the angle,** and the greater the amount of turning the greater the angle.

(*c*) Hence the **magnitude** of an angle depends upon the amount of revolution necessary to turn a line through the angle. The length of the sides of the angle bears no relation to the size of the angle. The magnitude of an angle may be made clear by means of a pair of dividers, the legs of the dividers representing the sides of the angle and the hinge, the vertex. If the dividers are opened a given amount, a certain angle is represented; if from that position they are closed more or less, a smaller angle is represented; if they are opened farther, a larger angle is represented.

(*d*) A line can turn through an angle in two directions, hence there are two angles which have the same sides and the same vertex.

For example, the side *O B*, in Fig. 8, can be made to turn through the angle *m* by a motion opposite to that of the hands of a watch, to the position *O A*, or it can be made to turn through the angle *n* by a motion like that of the hands of a watch, to the same position *O A*.

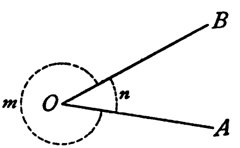


FIG. 8.

Two angles which have the same

sides and the same vertex are **conjugate angles.** When an angle is referred to, the smaller of the two conjugate angles is always meant, unless the other is specifically mentioned.

(*e*) The **direction of a line** is its position as determined by the angle it makes with a given line upon a given side of it. For instance, the direction northeast means a line which makes an angle of forty-five degrees with a north and south line on the east side of it.

A surveyor indicates a particular direction when he says "south $17^\circ\ 20'$ east." He means the line which makes an angle of $17^\circ\ 20'$ with the north and south line on the east side of it.

**18.** When the two sides of two conjugate angles lie in the same straight line, each conjugate angle is a **straight angle;** for example, the angle $A\ O\ B$ in Fig. 9 is a straight angle.

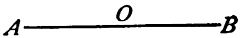



FIG. 9.

The two sides of a straight angle form a straight line.

**19.** Two angles which have a common vertex and one common side, and are on opposite sides of this common side, are **adjacent angles.** In Fig. 10, angles *1* and *2* are adjacent angles.

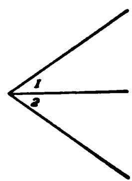



FIG. 10.

**20.** A **right angle** is an angle made by two straight lines which meet so that the adjacent angles formed are equal. In Fig. 10, if $A\ O$ and $B\ C$ meet so that the adjacent angles *1* and *2* are equal, angle *1* is a right angle; angle *2* is also a right angle.

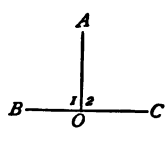



FIG. 11.

**21.** An **acute angle** is an angle that is less than a right angle. In Fig. 12, *B O C* is an acute angle.

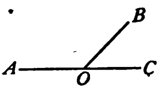


FIG. 12.

**22.** An **obtuse angle** is an angle that is greater than a right angle. In the same figure, *A O B* is an obtuse angle.

Acute and obtuse angles are **oblique angles.**

**23.** A **perpendicular line,** or a **perpendicular,** is one that makes right angles with another line. In Fig. 11, *A O* is perpendicular to *B C*, and *B C* is perpendicular to *A O*.

**24.** A line is **oblique** to another line when it makes oblique angles with that line. The two lines are sometimes called oblique lines. In the Fig. 12, *B O* is oblique to *A C*, and *A C* is oblique to *B O*.

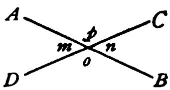


FIG. 13.

**25.** When two lines intersect, the opposite angles are **vertical angles.** The angles *m* and *n* are vertical angles; angles *p* and *o* are also vertical angles.

**26.** Two angles are **complementary** if their sum equals one right angle. The angles are then **complements** of each other.

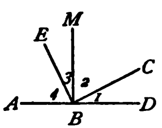


FIG. 14.

If in Fig. 14 *M B* is perpendicular to *A D*, angles *1* and *2* are complements of each other. If angles *A* and *B*, Fig. 15, are together equal to one right angle, they are complements of each other.

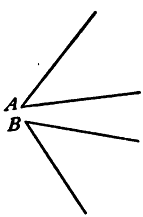


FIG. 15.

**27.** Two adjacent angles that are complements of each other are **complementary adjacent angles,** as angles *1* and *2*, Fig. 14.

**28.** Two angles are **supplementary** if their sum equals two right angles. The angles are then **supplements** of each other. Angle *1* and angle *A B C*, in Fig 14, and the angles *m* and *p*, in Fig. 13, are supplementary angles. If in Fig. 15 the sum of angles *A* and *B* is equal to two right angles, *A* and *B* are supplementary angles.

29. Two adjacent angles which are supplements of each other are **supplementary adjacent angles.**

## Logical Terms.

30. A **theorem** is a truth which requires demonstration. For example: *If two straight lines intersect each other, the vertical angles are equal.*

31. The statement of a theorem is its **enunciation,** or the **general enunciation.**

When a drawing is made to illustrate a theorem, the description of the drawing is the **special enunciation.**

32. A theorem consists of two parts, the **hypothesis** and the **conclusion.** The conditional part of a theorem is the **hypothesis.** For example, in the above theorem the hypothesis is: "*If two straight lines intersect each other.*" The hypothesis is sometimes called the **premises.**

The truth depending upon, or following from, the hypothesis is the **conclusion.**

In the theorem stated in article 30, the truth, "*the vertical angles are equal,*" depends upon the hypothesis, "*If two straight lines intersect each other,*" and is therefore the conclusion.

33. The **demonstration,** or **proof,** of a theorem is the course of reasoning by which the truth of the theorem is established.

34. A **problem** is a question proposed for solution, or the statement of certain relations which are to be pro-

duced. For instance: *To construct a line perpendicular to a given line at a given point.*

35. A **proposition** is a general term for a theorem or a problem.

36. A **corollary** is a proposition easily deduced from the proposition to which it is attached, with the aid, if necessary, of one or more previous propositions.

37. A **scholium** is a remark upon one or more propositions with respect to their applications, limitations, or connections.

38. An **axiom** is a truth which, from its simplicity, must be admitted without demonstration: as, *The whole of anything is equal to the sum of all its parts.*

39. A **postulate** is a proposition whose solution, on account of its simplicity, is admitted to be possible: as, *Let it be granted that a straight line can be drawn between two points.*

### Axioms.

40. *1.* Things which are equal to the same thing or equal things, are equal to each other.

*2.* If equals are added to equals the sums are equal.

*3.* If equals are subtracted from equals the differences are equal.

*4.* If equals are multiplied by the same number or by equals the products are equal.

COROLLARY.—Doubles of equals are equal.

*5.* If equals are divided by the same number or by equals the quotients are equal.

COROLLARY.—Halves of equals are equal.

*6.* If equals are added to unequals the sums are unequal, and that sum is the greater which is obtained by adding to the greater magnitude.

Corollary.—If unequals are multiplied by equals the products are unequal, and that product is the greater which is obtained by multiplying the greater magnitude.

*7.* If equals are subtracted from unequals the differences are unequal, and that difference is the greater which is obtained by subtracting from the greater magnitude.

Corollary.—If unequals are divided by equals, the quotients are unequal, and that quotient is the greater which is obtained by dividing the greater magnitude.

*8.* If unequals are subtracted from equals the differences are unequal, and that difference is less which is obtained by subtracting the greater magnitude.

*9.* The whole is greater than any of its parts.

*10.* The whole is equal to the sum of all its parts.

*11.* A straight line is the shortest distance between two points.

*12.* If two straight lines have two points in common they are one and the same straight line.

Corollary 1.—Two straight lines can intersect in but one point.

Corollary 2.—But one straight line can be drawn between two points.

*13.* Magnitudes which coincide are equal in all respects, and conversely, magnitudes which are equal may be made to coincide.

*14.* Every magnitude has two halves or is equal to two halves of itself.

Corollary.—Every magnitude, however small, may be divided into two or more parts.

*15.* If any magnitude, however small, is added to one of two equal magnitudes, and the same subtracted from the other, the results are unequal.

*16.* There is only one shortest line between two points or between a point and a line.

*17.* A magnitude which is less or greater than one of two equal magnitudes is less or greater, respectively, than the other.

*18.* Only one line can be drawn through a given point parallel to a given line.

POSTULATE 1.—A line can be revolved about a given point until it embraces another point or takes the direction of a line drawn through the given point.

POSTULATE 2.—A figure can be thought of as being changed in position without making any change in the relation of its parts.

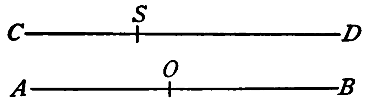
**Symbols and Abbreviations.**

| | |
|---|---|
| **41.** ∠ —*angle* | Ax.—*axiom.* |
| ∠s—*angles* | Cons.—*construction.* |
| ⊥ —*perpendicular* | Cor.—*corollary.* |
| ⊥s—*perpendiculars* | Def.—*definition.* |
| ∥ —*parallel* | Ex.—*exercise.* |
| ∥ s—*parallels* | Auth.—*authority.* |
| △ —*triangle* | Hyp.—*hypothesis.* |
| △s—*triangles* | Rt.—*right.* |
| ▱ —*parallelogram* | Sch.—*scholium.* |
| ▱s—*parallelograms* | St.—*straight.* |
| ⊙ —*circle* | Sug.—*suggestion.* |
| ⊙s—*circles* | Sugs.—*suggestions.* |
| ∴ —*therefore.* | P.—*Postulate.* |

Q. E. D.—*Which was to be demonstrated.*

### Proposition I.*

**42. Theorem.** *Two straight angles are equal.*




**Let A O B and C S D represent two straight angles.**

*To prove that angle A O B is equal to angle C S D.*

*Suggestion 1.* What kind of a line do the two sides of $\angle AOB$ form? Of $\angle CSD$? § 18.

*2.* Place $\angle AOB$ so that point $O$ lies upon $S$, and another point of line $AOB$ upon line $CSD$. § 40, P. 2.

*3.* Where does $\angle AOB$ lie? *Ax.* 12.

*4.* How then does the amount of revolution in turning through $\angle AOB$ (§ 17 *c*), compare with that in turning through $\angle CSD$? *Ax.* 13.

*Therefore*—

What is the hypothesis in this theorem?

What is the conclusion?

What is the general enunciation? The special?

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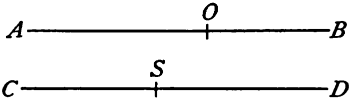
Ex. 1. Given three points not in a straight line, how many straight lines can be drawn through them, each line being drawn through two points?

* Note.—The pupil is expected to study the suggestions carefully, to follow the directions when directions are given, to answer the questions when questions are asked, and to give the authority on which the answers are based; then to review the whole demonstration in a consecutive manner, without the aid of the suggestions. To illustrate what is expected of the pupil, model demonstrations are given of a few propositions, but no model should be consulted until after the proposition has been studied by means of the suggestions. The pupil will be more likely to avoid indefiniteness if he writes out all of his demonstrations for the first few weeks.

## Model.

### PROPOSITION I.

**Theorem.** *Two straight angles are equal.*




**Let A O B and C S D represent two straight angles.**

*To prove that angle A O B is equal to angle C S D.*

The two sides of $\angle A O B$ and $\angle C S D$ form straight lines. § 18.

Place $\angle A O B$ upon $\angle C S D$ so that point $O$ lies upon $S$ and some other point of line $A O B$, as $A$, lies upon a part of line $C S D$. § 40, P. 2.

The lines $A O B$ and $C S D$ coincide, and therefore are one and the same line. *Ax.* 12.

$\therefore$ The $\angle A O B = \angle C S D$. *Ax.* 13, § 17 c.

*Therefore*—Two straight angles are equal. Q. E. D.

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Ex. 2. Given four points, no three of which are in the same straight line, how many straight lines can be drawn through them if each line connects two of the four points?

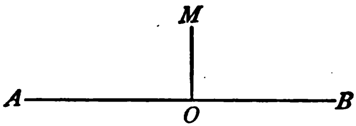
Ex. 3. What is the greatest number of points in which three straight lines can intersect?

Ex. 4. What is the greatest number of points in which four straight lines can intersect?

Ex. 5. If an angle is a right angle, what is its supplement?

## PROPOSITION II.

**43. Theorem.** *One straight angle is equal to two right angles.*




**Let A O B represent a straight angle.**

*To prove that the angular magnitude A O B is equal to two right angles.*

*Suggestion 1.* What kind of a line is *A O B*? § 18.

*2.* Let *M O* represent a straight line that makes equal angles with the line *A B*. *Ax.* 14.

What kind of ∠s are *A O M* and *M O B*? § 20.

*Therefore—*

**44.** A perpendicular is **erected to a line** when it is drawn perpendicular to the line from a point in the line.

**45.** A perpendicular is **dropped to a line** when it is drawn perpendicular to the line from a point without the line.

---

Ex. 6. If an angle is two-thirds of a right angle, what is its supplement?

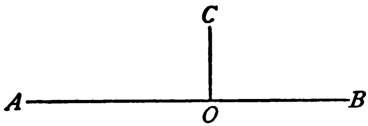
Ex. 7. If an angle is three-fourths of a right angle, what is its complement?

Ex. 8. The complement of angle $x$ equals one-third of its supplement. Find what part of a right angle $x$ is.

Ex. 9. The supplement of angle $x$ is two and one-half times its complement. Find $x$, the complement, and the supplement, each in terms of a right angle.

## PROPOSITION III.

**46. Theorem.** ***At any point in a straight line one perpendicular, and but one, can be erected.***



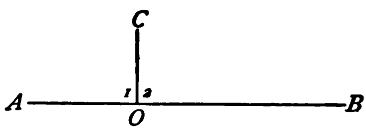

**Let A B represent a straight line and O any point in the line.**

FIRST.—*To prove that a perpendicular can be erected to A B at O.*

*Suggestion 1.* Draw *C O* to meet *A B* at *O*, making the two adjacent angles equal. *Ax.* 14.

2. See § 20 and 23.

SECOND.—*To prove that but one perpendicular can be erected to A B at O.*




*Suggestion 1.* Represent a perpendicular to *A B* at *O*, as *C O*. § 46, Part I.

2. How would ∠s *1* and *2* compare, if *C O* should be revolved either way, however little? Why? Ax. 15.

3. ∠s *1* and *2* would then be what kind of ∠s? Auth.

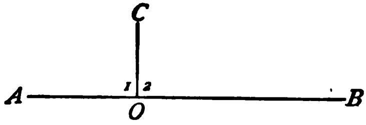
4. *C O* would be what kind of line? § 24.

*Therefore—*

### Model.

## Proposition III.

**Theorem.** *At any point in a straight line one perpendicular and only one can be erected.*




**Let AB represent a straight line and O any point in that line.**

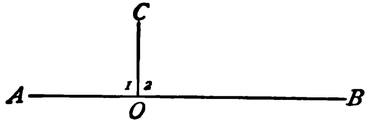
First.—*To prove that a perpendicular can be erected to A B at O.*

Let *C O* represent a line making two equal adjacent ∠s with *A B* at *O*, as ∠ *1* and ∠ *2*. *Ax.* 14.

∠s 1 and 2 are right ∠s. § 20.

*C O* is ⊥ to *A B*. § 23.

*Therefore*—At a given point in a line a perpendicular can be erected to the line. Q. E. D.




Second.—*To prove that but one perpendicular can be erected to A B at O.*

**Let C O represent a perpendicular to A B at O.**

If $C O$ is revolved ever so little about the point $O$, $\angle$ 1 would be greater or less than $\angle$ 2. *Ax.* 15.

Hence $\angle$s 1 and 2 would be oblique $\angle$s. § 22.

$C O$ would be an oblique line. § 24.

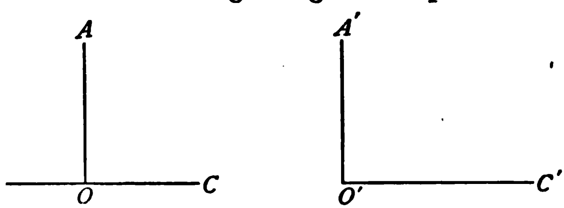
*Therefore*—Not more than one perpendicular can be erected to a line at a given point. Q. E. D.

47. COROLLARY.—*Through the vertex of a given angle one and only one straight line which bisects the given angle can be drawn.*

*Suggestion.*—See method used in demonstrating the theorem. *Axs.* 14 and 15.

## PROPOSITION IV.

48. **Theorem.** *All right angles are equal.*




**Let A O C and A′ O′ C′ represent any two right angles.**

*To prove that angle $A\ O\ C$ and angle $A'O'C'$ are equal.*

*Suggestion* 1. Place $\angle A\ O\ C$ upon $\angle A'O'C'$ so that $A\ O$ lies upon $A'O'$, $O$ upon $O'$. § 40, P. 2.

2. Where does $O\ C$ fall? § 46.

3. How then do $\angle A\ O\ C$ and $\angle A'O'C'$ compare? *Ax.* 13.

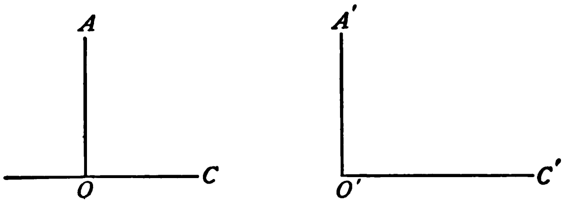
49. Any magnitude is **bisected** when it is divided into two equal parts.

50. Two figures **coincide** when each point of one lies in a corresponding point of the other.

**Model.**

PROPOSITION IV.

**Theorem.** *All right angles are equal.*




**Let A O C and A′ O′ C′ represent any two right angles.**

*To prove that angle $A\ O\ C$ is equal to angle $A'O'C'$.*

Place $\angle A\ O\ C$ upon $\angle A'O'C'$, $A\ O$ upon $A'O'$ with $O$ upon $O'$. § 40, P. 2.

Line $O\ C$ falls upon line $O'C'$. § 46.

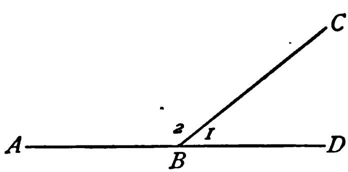
$\angle A\ O\ C$ coincides with $\angle A'O'C'$. § 50.

∴ Rt. $\angle A\ O\ C =$ Rt. $\angle A'O'C'$. *Ax.* 13. § 17 (c).

*Therefore*—All right angles are equal. Q. E. D.

PROPOSITION V.

51. **Theorem.** *If one straight line meets another straight line, the sum of the two adjacent angles formed is equal to two right angles.*




**Let C B and A D represent any two straight lines which meet at a point, as B, forming two adjacent angles, as angles 1 and 2.**

*To prove that the sum of angles 1 and 2 is equal to two right angles.*

FIRST.—Suppose ∠s *1* and *2* are equal.

*Suggestion 1.* How many right angles are formed? § 20.

SECOND.—Suppose ∠s *1* and *2* are unequal.

*1.* What kind of ∠ is *A B D?* § 18.

*2.* Compare ∠ *1* + ∠ *2* with ∠ *A B D.* 17 c, *Ax.* 13.

*3.* See Proposition II. and complete the demonstration.

*Therefore—*

QUERY.—In Proposition V., what is the hypothesis? What is the conclusion?

**52.** COROLLARY 1.—*The sum of all the angles on one side of a straight line, having a common vertex in the line, is equal to two right angles.*

**53.** COROLLARY 2.—*The total angular magnitude about a point is equal to four right angles.*

---

Ex. 10. If the angular magnitude about a point is divided into six equal angles, each angle is what part of a right angle?

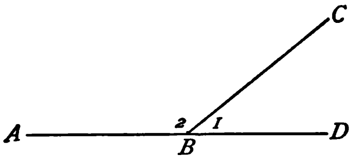
Ex. 11. If the angular magnitude about a point is divided into three angles, the second of which is twice the first, and the third is three times the first, how many right angles in each of the three angles?

Ex. 12. A line drawn perpendicular to the bisector of an angle at the vertex makes equal angles with the sides of the angle.

### Model.

### Proposition V.

**Theorem.** *If one straight line meets another straight line, the sum of the two adjacent angles formed is equal to two right angles.*




**Let C B meet A D, at B.**

*To prove that the sum of angles 1 and 2 is equal to two right angles.*

First.—If $\angle$s 1 and 2 are equal.

$\angle 1 + \angle 2 = 2$ Rt. $\angle$s. § 20.

Second.—If $\angle$s 1 and 2 are unequal.

$\angle 1 + \angle 2 =$ the st. $\angle A B D$. § 17 (c), *Ax.* 13.

The st. $\angle A B D = 2$ Rt. $\angle$s. § 43.

$\therefore \angle 1 + \angle 2 = 2$ Rt. $\angle$s. *Ax.* 1.

*Therefore*—If one straight line meets another straight line, the sum of the two adjacent angles is equal to two right angles.

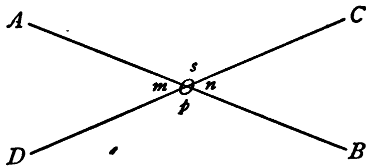
---

Ex. 13. In Fig. 12, if angle $C O B$ is one-half of a right angle, angle $A O B$ equals what?

Ex. 14. In Fig. 14, if angle 1 is one-third of a right angle, and angle 4 is three-fourths of a right angle, angle $C B E$ equals what?

## PROPOSITION VI.

**54. Theorem.** *If two straight lines intersect, the vertical angles formed are equal.*




**Let A B and C D intersect at O, forming the vertical angles *m* and *n*, and *s* and *p*.**

*To prove that angle* ***m*** *is equal to angle* ***n***.

*Suggestion 1.* $\angle m + \angle s =$ what? Why?

*2.* $\angle s + \angle n =$ what? Why?

*3.* Compare $\angle m + \angle s$ and $\angle n + \angle s$. Give auth.

*4.* Compare $\angle m$ and $\angle n$. *Ax.* 3.

*Therefore*—

In a similar manner compare $\angle s$ and $\angle p$.

---

Ex. 15. If there are three angles about a point, and one of them is equal to one and one-fifth right angles, and one to nine-tenths of a right angle, what is the magnitude of the other?

NOTE.—The student should make a careful study of the form and nature of a demonstration. In respect to form: first in order is the statement of the theorem or the **general enunciation**, this should be followed by the application of the theorem to a figure or the **special enunciation**, then follows the **proof**, and finally the **conclusion**.

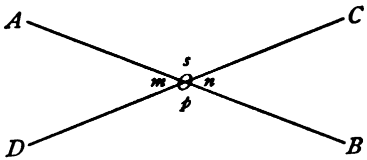
In the special enunciation each feature of the theorem should be carefully applied, close attention being paid, first, to what is given or known, and second, to what is to be determined. In the proof, each statement made should be based upon authority, which should consist of an axiom, a postulate, a definition, or a previously proved proposition. The student should stand ready to demonstrate all propositions used as authority. Care should be taken to see that these authorities exactly apply.

The order of procedure in an original demonstration depends upon the use that can be made of what is given, in arriving at what is to be determined.

### Model.

PROPOSITION VI.

**Theorem.** *If two straight lines intersect, the vertical angles formed are equal.*




**Let AB and CD intersect at O, forming the vertical angles *m* and *n*.**

*To prove that angle m equals angle n.*

$\angle m + \angle s = 2$ Rt. $\angle$s. § 51.

$\angle n + \angle s = 2$ Rt. $\angle$s. § 51.

$\therefore \angle m + \angle s = \angle n + \angle s$. *Ax.* 1.

$\therefore \angle m = \angle n$. *Ax.* 3.

QUERY.—In this proposition, which is the special enunciation? Which the general? Which the hypothesis? Which the conclusion?

### Polygons.

**55.** A **polygon** is a portion of a plane bounded by straight lines; as *M N O*, etc., in Fig. 1.

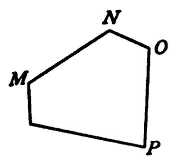



FIG. 1.

**56.** The bounding lines are the **sides** of the polygon, and their sum is the **perimeter** of the polygon.

**57.** The angles formed by the sides of the polygon on the side of the inclosed space, are the **interior angles** of the polygon, as angle *B A E* in Fig. 2.

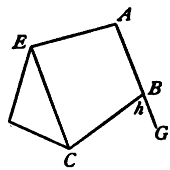



FIG. 2.

58. An angle formed by one side of the polygon and an adjacent side extended, is an **exterior angle** of the polygon; as angle *h* in Fig. 2.

59. The vertices of the interior angles of a polygon are the **vertices of the polygon.**

60. A straight line joining any two vertices, not adjacent, is a **diagonal** of a polygon; as *E C* in Fig. 2.

61. If there is no ambiguity, a polygon may be **read** by naming any two vertices not adjacent; as *A D*, *B D*, *M P*, etc.

62. Polygons are classified according to the **number of their sides.** The least number of sides a polgyon can have is three.

63. A polygon of three sides is a **triangle.**

64. A polygon of four sides is a **quadrilateral.**

65. A polygon of five sides is a **pentagon.**

66. A polygon of six sides is a **hexagon,** etc.

67. An **equilateral polygon** is a polygon all of whose sides are equal.

68. An **equiangular polygon** is a polygon all oı whose angles are equal.

69. A **convex polygon** is a polygon no side of which, if extended, enters the space inclosed by the perimeter of the polygon; as *A B C*, etc., Fig. 2.

70. A **concave polygon** is a polygon, two or more sides of which, if extended, would enter the space inclosed by the perimeter of the polygon; as Fig. 3. If either *A B* or *B C* is extended through *B* it would enter the space inclosed by the perimeter of the polygon.

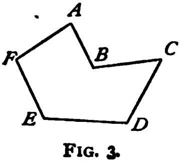


FIG. 3.

71. The angle $A\,B\,C$ in this figure is a **re-entrant angle.**

72. A **right angled triangle,** or a **right triangle,** is a triangle one of whose angles is a right angle.

(a) The side opposite the right angle is the **hypotenuse.**

(b) The other two sides are **the legs** of the triangle.

73. An **acute angled triangle,** or an **acute triangle,** is a triangle all of whose angles are acute.

74. An **obtuse angled triangle,** or an **obtuse triangle,** is a triangle one of whose angles is obtuse.

75. Acute or obtuse triangles are sometimes called **oblique triangles.**

76. A **scalene triangle** is a triangle no two sides of which are equal.

77. An **isosceles triangle** is a triangle which has two equal sides. The equal sides are the **legs** of the triangle.

78. A triangle which has all three sides equal is an **equilateral triangle.**

79. The **base** of a triangle is a selected side or the side upon which it is supposed to stand. The angle which is opposite the base is the **vertical angle** or the **vertex** of the triangle. Generally, any side may be taken as the base, but in an isosceles triangle that side which is not one of the two equal sides is always considered the base.

80. The **altitude** of a triangle is the perpendicular from the vertex to the base or the base extended.

NOTE.—The word altitude may refer to the line or to the length of the line, expressed in terms of some unit. The context will determine which use is intended.

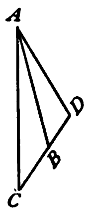
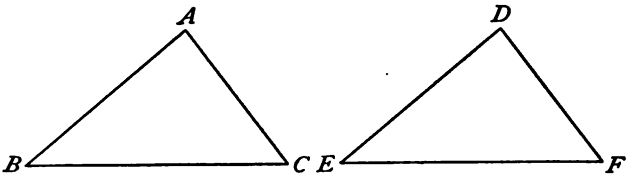


FIG. 4.

81. The **median line,** or **median** of a triangle, is a line drawn from the vertex to the middle of the opposite side. $A\,B$ is the median, in Fig. 4.

## PROPOSITION VII.

82. **Theorem.** *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*




**Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *A B C* upon △ *D E F*, so that point *B* is upon *E* and line *B C* lies in *E F*. § 40 P.

Where does point *C* fall? Why?

2. What direction does *B A* take with respect to *E D?* § 17 (e).

3. Where does the point A lie? Why?

4. Points *C* and *A* being located, where, with respect to *D F*, does the line *A C* lie? *Ax.* 12, Cor. 11.

5. What, now, is the position of △ *A B C* with respect to △ *D E F?* § 50.

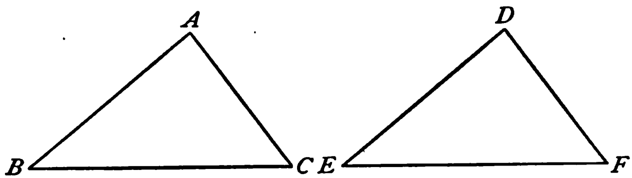
6. How, then, does △ *A B C* compare with △ *D E F*? *Ax.* 13.

*Therefore—*

## Model.

### Proposition VII.

**Theorem.** *If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles are equal in all respects.*




**Let A B C and D E F represent two triangles, in which A B is equal to D E, B C is equal to E F, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

Place $\triangle\ ABC$ upon $\triangle\ DEF$, so that $B$ is upon $E$ and $BC$ lies in $EF$. § 40, P. 1.

$C$ falls upon $F$. $(BC = EF.)$

$BA$ takes the direction of $ED$. $(\angle B = \angle E.)$ § 17 (e).

$A$ falls upon $D$. $(BA = ED.)$

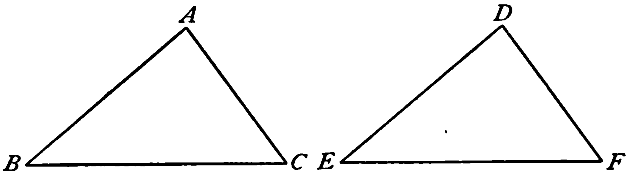
$AC$ coincides with $DF$. *Ax.* 12 Cor. 2.

$\therefore \triangle\ ABC = \triangle\ DEF$. *Ax.* 13. Q. E. D.

NOTE.—The method of proof in this proposition is the method of superposition, and consists in mentally placing one figure upon the other and finding that they exactly coincide.

## PROPOSITION VIII.

**83. Theorem.** *If two triangles have two angles and the included side of one, equal to two angles and the included side of the other, each to each, the triangles are equal in all respects.*




**Let ABC and DEF represent two triangles, having BC equal to EF, angle B equal to angle E, and angle C equal to angle F.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *A B C* upon △ *D E F*, so that *B* falls upon *E*, and *B C* on *E F*. Where does *C* fall? Why?

*2.* What direction does *B A* take? Why?

*3.* Where does *A* fall? Why? Sug. 2.

*4.* What direction does *C A* take? Why?

*5.* Where, now, does the point *A* fall? Sugs. 2 and 4.

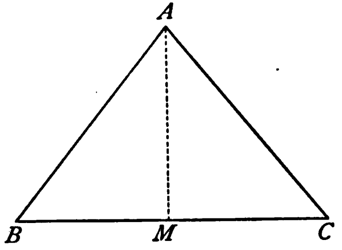
*6.* Then how does △ *A B C* compare with △ *D E F?* *Ax.* 13.

*Therefore* —

84. SCHOLIUM. In equal triangles, equal angles lie opposite equal sides, and equal sides lie opposite equal angles.

## PROPOSITION IX.

**85. Theorem.** *The angles opposite the equal sides of an isosceles triangle are equal.*




**Let ABC represent an isosceles triangle, AB being equal to AC.**

*To prove that angle B is equal to angle C.*

*Suggestion 1.* Let *A M* be drawn to represent a bisector of ∠ *A*, and be extended until it meets *B C*, as at *M*.

*2.* Compare △ *A B M* with △ *A C M*. § 82.

*3.* Compare ∠ *B* with ∠ *C*. § 84.

*Therefore* —

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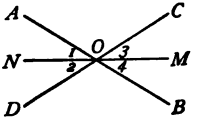
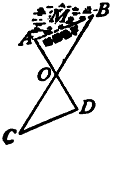



Ex. 16. If a straight line bisects one of a pair of vertical angles, prove that it bisects the other also.

If *M N* bisects ∠ *A O D*, prove that it bisects ∠ *C O B*; that is, that ∠ 3 = ∠ 4.

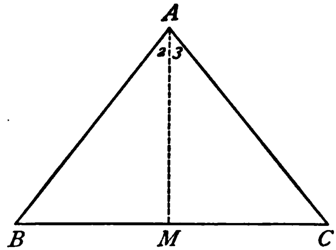



Ex. 17. Let *M* represent a swamp or pond. Required to find the distance *A B*.

## Model.

### Proposition IX.

**Theorem.** *The angles opposite the equal sides of an isosceles triangle are equal.*




**Let ABC represent an isosceles triangle, AB being equal to AC.**

*To prove that angle B is equal to angle C.*

Let $AM$ be drawn to represent the bisector of $\angle A$, and be extended to meet $BC$, as at $M$.

In $\Delta$s $ABM$ and $ACM$,

| | |
|---|---|
| $AB = AC.$ | Hyp. |
| $\angle 2 = \angle 3.$ | Cons. |
| $AM = AM.$ | *Ax.* 13. |
| $\therefore \Delta ABM = \Delta ACM.$ | § 82. |
| *Therefore:* $\angle B = \angle C.$ | § 84. |

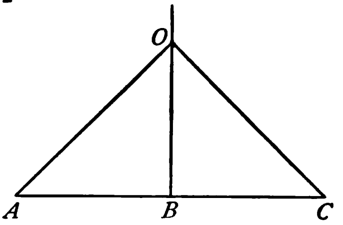
Q. E. D.

---

Ex. 18. If the equal sides of an isosceles triangle are extended beyond the base, prove that the exterior angles formed with the base are equal.

## PROPOSITION X.

**86. Theorem.** *If a perpendicular be erected at the middle point of a straight line, the distances from any point in the perpendicular to the extremities of the line are equal.*



**Let A C represent any line, B its middle point, B O a perpendicular to A C at B, and O any point in the perpendicular.**

*To prove that O A is equal to O C.*

*Suggestion 1.* In $\triangle$s *O B A* and *O B C*, what parts are equal, each to each? Why?

*2.* How do $\triangle$s *O B A* and *O B C* compare? § 82.

*3.* How, then, do *O A* and *O C* compare? § 84.

*Therefore* —

---

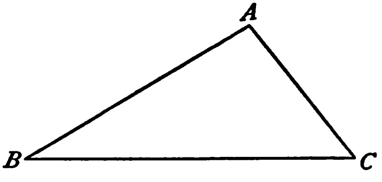
Ex. 19. Points, in the sides of an isosceles triangle, equidistant from the extremities of the base, are equidistant from the vertex.

Ex. 20. Prove that the line which bisects the vertical angle of an isosceles triangle bisects the triangle.

Ex. 21. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base. §s 23 and 20.

## PROPOSITION XI.

87. **Theorem.** ***Any side of a triangle is less than the sum of the other two.***



**Let A B C represent any triangle.**

*To prove that any side, as A B, is less than the sum of the other two.*

*Suggestion:* Which represents the shorter distance from *A* to *B*, that by way of the line *A B*, or that by way of the lines *A C* and *C B?* Why? *Ax.* 11.

*Therefore* —

Ex. 22. If two straight lines intersect and one of the angles formed is a right angle, all of the angles are right angles.

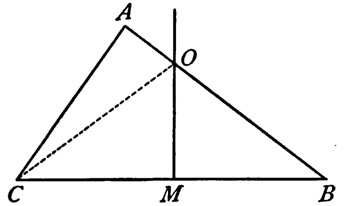
Ex. 23. A line which is perpendicular to the bisector of an angle makes equal angles with the sides of the angle. (1) if drawn through the vertex (Ex. 12); (2) if drawn through any other point of the bisector of the angle.

Ex. 24. Prove that an equilateral triangle is also equiangular.

Ex. 25. If the middle points of the sides of an isosceles triangle be joined by straight lines another isosceles triangle is formed. § 82.

## Proposition XII.

**88. Theorem.** *If a perpendicular be erected at the middle point of a straight line, the distances from a point not in the perpendicular to the extremities of the line are unequal.*




**Let B C represent any straight line, M its middle point, O M a perpendicular to B C at the point M, A any point not in the perpendicular, and A B and A C lines drawn from A to the extremities of the line B C.**

*To prove that A B and A C are unequal.*

*Suggestion 1.* Let $O$ be the intersection of $A B$, and the perpendicular. Draw the line $O C$.

2. Compare $A C$ with $A O + O C$. § 87.
3. Compare $O C$ with $O B$. § 86.
4. Compare $O C + O A$ with $O B + O A$. *Ax.* 2.
5. Compare $A C$ with $A B$. *Ax.* 17.

*Therefore* —.

Query: Why draw the line $O C$?

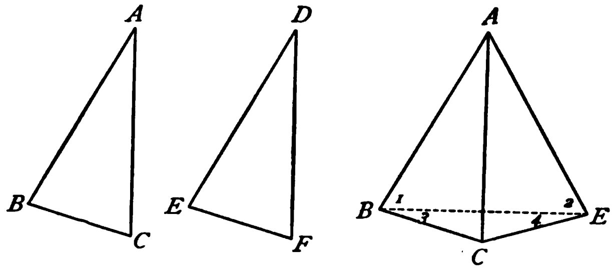
Draw the figure so that $A C$ crosses $O M$, and prove the proposition.

---

Ex. 26. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other. *Ax.* 5.

## Proposition XIII.

**89. Theorem.** *Two triangles, having the three sides of one equal, respectively, to the three sides of the other, are equal in all respects.*




**Let ABC and DEF represent two triangles, having AB equal to DE, AC equal to DF, and BC equal to EF.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place △ *D E F* upon △ *A B C*, so that the longest side, *D F*, of △ *D E F*, coincides with the longest side, *A C*, of △ *A B C*, *D* upon *A*, and *F* upon *C*, but the point *E* upon the opposite side of *A C*, from *B*. Draw *B E*.

2. In △ *A B E*, compare *A B* with *A E*. Give auth.
3. Compare ∠ 1 with ∠ 2. § 85.
4. In △ *C B E*, compare ∠ 3 with ∠ 4. Give auth.
5. Compare ∠ *B* with ∠ *E*. *Ax.* 2.
6. Compare △ *A B C* with △ *A E C*. § 82.
7. Then how does △ *A B C* compare with △ *D E F?*

*Therefore —*

**90.** The **premises** of a proposition are the conditions given upon which the conclusion is based.

91. The **converse** of a given proposition is a proposition which has the conclusion of the given proposition for one of the premises, and a premise of the given proposition for the conclusion.

Proposition *XIV.*, separated into premises and conclusion, may be stated:

*Premises:* The sum of two adjacent angles is equal to two right angles.

*Conclusion:* The exterior sides form a straight line.

Proposition *XIV.*, is the converse of Proposition *V.*, which, separated into premise and conclusion and interpreted in the language of Proposition *XIV.*, may be stated:

*Premises:* Two adjacent angles have a straight line for their exterior sides.

*Conclusion:* The sum of the angles is equal to two right angles.

92. When a proposition is proved to be true, it does not necessarily follow that its converse is also true.

---

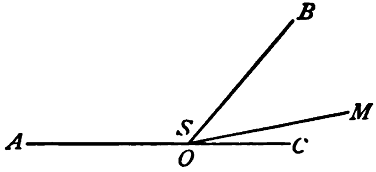
Ex. 27. If $A B C$ is an equilateral triangle, and $D$, $E$, and $F$ are points in the sides $A B$, $B C$, and $C A$, respectively, such that $A D$ is equal to $B E$ and to $C F$, prove that triangle $D E F$ is equilateral.

Ex. 28. If the straight line that joins the vertex of a triangle with the middle point of the base is perpendicular to the base, the triangle is isosceles.

Ex. 29. Prove that a line drawn from the vertex of an isosceles triangle to the middle of the base, (*1*) bisects the triangle, (*2*) bisects the vertical angle, (*3*) is perpendicular to the base.

## PROPOSITION XIV.

93. **Theorem.** *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*




**Let OA, OB, and OC be any three straight lines, which meet to form two adjacent angles S and BOC, whose sum is equal to two right angles.**

*To prove that the exterior sides, O A and O C, form a straight line.*

*Suggestion 1.* $AOC$ is either a straight line or a broken line. To determine which of these suppositions is true, represent an extension of $OA$, as $OM$.

*2.* How many rt. $\angle$s in $\angle S + \angle BOM$? Why?

*3.* How many rt. $\angle$s in $\angle S + \angle BOC$? Why?

*4.* Compare the sum of $\angle S + \angle BOM$, with the sum of $\angle S + \angle BOC$. Give auth.

*5.* Compare $\angle BOC$ with $\angle BOM$. Give auth.

*6.* Since the equal $\angle$s $BOC$ and $BOM$ have the common vertex $O$, and the common side $OB$, and since $OC$ and $OM$ are on the same side of $OB$, where does $OC$ lie with respect to $OM$? Why?

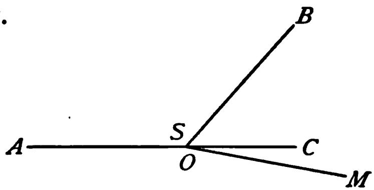
*7.* $OM$ in an extension of $OA$. Cons.
What relation does $OC$ sustain to $OA$?

*Therefore* —

## Model.

### Proposition XIV.

**Theorem.** *If the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*




**Let OA, OB, and OC, be any three straight lines, which meet to form two adjacent angles, S and BOC, whose sum is equal to two right angles.**

*To prove that the exterior sides, O A and O C, form a straight line.*

The line *A O C* is either a straight line or a broken line.

Draw *O M* an extension of *O A*.

$\angle S + \angle BOM = 2$ rt. $\angle$s. § 51.

$\angle S + \angle BOC = 2$ rt. $\angle$s. Hyp.

$\therefore \angle S + \angle BOM = \angle S + \angle BOC.$ *Ax.* 1.

$\therefore \angle BOM = \angle BOC.$ *Ax.* 3.

Since the equal $\angle$s *B O M* and *B O C* have a common vertex and the common side *O B*, and since *O C* and *O M* are on the same side of *O B*, *O C* must lie upon *O M*. § 17 (e).

*A O M* is a straight line. Cons.

*O C* lies upon *O M*.

$\therefore$ *A O C* is a straight line.

*Therefore*—If the sum of two adjacent angles equals two right angles, their exterior sides form a straight line.

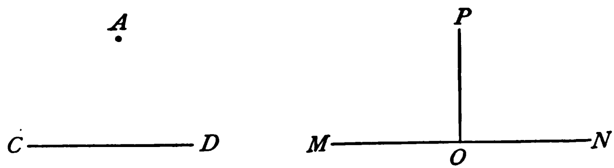
Ex. 30. Two angles are complements of each other, and the greater exceeds the less by 38 degrees. What are the angles?

Ex. 31. If two straight lines bisect each other at right angles, any point of either is equidistant from the extremities of the other. § 86.

Ex. 32. In the isosceles triangle *A B C*, *M* and *N* are in the base *B C*, so that angles *B A M* and *C A N* are equal. Prove triangle *B A M* is equal to *C A N*.

## Proposition XV.

94. **Theorem.** *One perpendicular, and only one, can be dropped from a point to a line.*




**Let C D represent any straight line, and A any point without the line.**

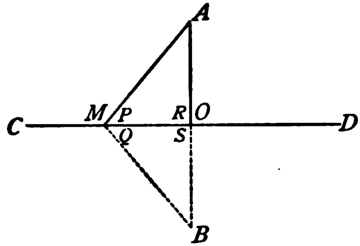
Case I. *To prove that one perpendicular can be dropped from A to the line C D.*

*Suggestion 1.* Draw any straight line *M N*, and at any point of this line *O* erect the ⊥ *O P*. § 46.

2. Place the line *M N* upon the line *C D* and move it back and forth in *C D*. *O P* must at some time embrace point *A*.

Then *O P* is ⊥ to *C D* from point *A*.

*Therefore* —




**Let A O represent one perpendicular dropped from the point A to the line C D.** § 94 Case I.

Case II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

*Suggestion 1.* If another ⊥ can be dropped, let it be represented by $A M$.

*2.* Extend $O A$ to $B$, making $O B = O A$, and connect $M$ and $B$.

*3.* Compare $\triangle A O M$ with $\triangle B O M$. Give auth.

*4.* Compare $\angle P$ with $\angle Q$. Give auth.

*5.* If by construction $\angle P$ is a rt. ∠, what is $\angle Q$? Is line $A M B$ a straight or a broken line? Why? § 93.

*6.* Then how many straight lines are drawn from $A$ to $B$?

*7.* What, then, do you conclude about the statement that $A M B$ is a straight line? Why? *Ax.* 12, Cor. 2.

*8.* What do you conclude as to the possibility of $A M$ being a ⊥ from $A$ to $C D$? Why? Sug. 1.

*9.* Then, how many ⊥s can be dropped from a point to a straight line?

*Therefore* —

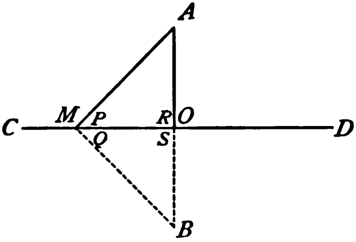
---

Ex. 33. The supplement of an acute angle is how much more than the complement of the same acute angle?

**Model.**

PROPOSITION XV.

**Theorem.** *Only one perpendicular can be dropped from a given point to a line.*




**Let O A represent one perpendicular dropped from the point A to the line C D.**

CASE II. *To prove that no other perpendicular can be dropped from the point A to the line C D.*

If another ⊥ can be dropped let it be represented by *A M.*

Extend *O A* to *B*, so that *O A* = *O B*, and connect *M* and *B*.

In △ *A O M* and *B O M*, *O A* = *O B*. Cons.

*O M* = *O M*. Identical.

∠ *R* = ∠ *S*. § 48.

∴ △ *B O M* = △ *A O M*. § 82.

∴ ∠ *P* = ∠ Q. § 84.

Since by hypothesis *P* is a rt. ∠, *Q* is a rt. ∠. *Ax.* 1.

∴ *A M B* is a straight line. § 93.

But *A O B* is a straight line by construction.

∴ *A M B* is a broken line. *Ax.* 12, Cor. 2.

∴ *A M* is not ⊥ to *C D*.

∴ But one ⊥ can be dropped from a point to a line.

Q. E. D.

**95.** It is often convenient to express the magnitude of an angle in some other way than by using a right angle as the unit. To obtain another unit a right angle is divided into ninety equal parts, called **degrees.** The magnitude of an angle may, then, be expressed by stating how many degrees the given angle contains.

**96. Theorem.** *Complements of equal angles are equal.*

**97. Theorem.** *Supplements of equal angles are equal.*

NOTE.—In elementary Geometry only acute angles have complements, but either acute or obtuse angles may have supplements.

---

Ex. 34. If one of two supplementary adjacent angles is bisected, a perpendicular to the bisector through the vertex bisects the other angle.

Ex. 35. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplements of each other.

Ex. 36. How many degrees in a straight angle? In all the angular magnitude about a point?

Ex. 37. How many degrees in the supplement of two-thirds of a right angle?

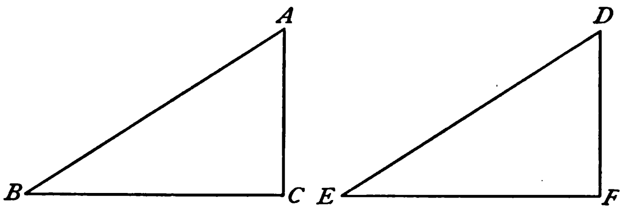
Ex. 38. How many degrees in an angle whose complement equals one-fourth of its supplement?

Ex. 39. The supplement of ten degrees is how much more than the complement of ten degrees?

Ex. 40. The straight lines bisecting the equal angles of an isosceles triangle and terminating in the sides, are equal. § 83.

## PROPOSITION XVI.

98. **Theorem.** *Two right triangles which have the hypotenuse and an adjacent angle of one, equal to the hypotenuse and an adjacent angle of the other, each to each, are equal in all respects.*




**Let A B C and D E F represent two right triangles, having the hypotenuse A B equal to the hypotenuse D E, and angle A equal to angle D.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place $\triangle$ *A B C* upon $\triangle$ *D E F*, so that *A B* coincides with *D E*, *A* upon *D*, and *B* upon *E*.

2. What direction does *A C* take? Why? § 17 (e).

3. Where does the point *C* fall? Why?

4. Since *B C* and *E F* are both $\perp$ to the line *D F*, from point *E* where does *B C* lie? Why? § 94.

5. Where does point *C* fall? Why? Sugs. 2 and 4.

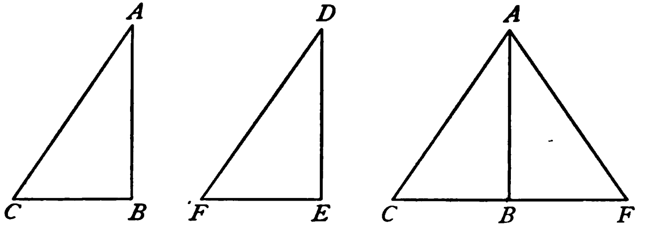
6. How, then, do the two $\triangle$s compare? Why?

*Therefore* —

NOTE.—In the answer to suggestion 3, it will be seen that *C* must lie somewhere in the line *D F*, and, in the answer to Suggestion 5, *C* must lie somewhere in the line *E F*. Hence, in the answer to Suggestion 5, *C* can be exactly located.

PROPOSITION XVII.

**99. Theorem.** *Two right triangles which have the hypotenuse and a side of one, equal to the hypotenuse and a side of the other, each to each, are equal in all respects.*




**Let A B C and D E F represent two right triangles, having the hypotenuse A C equal to the hypotenuse D F, and the side A B equal to the side D E.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion 1.* Place the triangles so that *A B* coincides with *D E*, *A* upon *D*, and *B* upon *E*, with the vertices, *C* and *F*, on opposite sides of *A B*.

*2.* Is *C B F* a straight or broken line? Why? § 93.

*3.* What kind of an Δ is *A C F?* Why?

*4.* Compare ∠s *C* and *F*. § 85.

*5.* Compare Δs *A B C* and *A B F*, *A B C* and *D E F*.

*Therefore —*

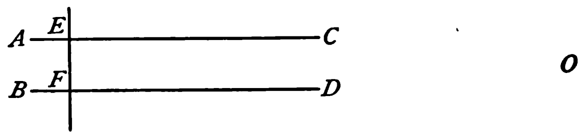
**100.** Name all the methods of determining the equality of triangles that have been demonstrated. Name those that relate to right triangles.

How many parts of a triangle must be equal in order to make the triangles equal? Suppose the three angles of one triangle equal the three angles of another triangle, each to each, how do the triangles compare in equality?

**101.** Straight lines, in the same plane, that do not and cannot meet, however far extended, are **parallel lines.**

## PROPOSITION XVIII.

**102. Theorem.** *Two lines, which are perpendicular to the same line, are parallel.*




**Let A C and B D represent two lines, each perpendicular to the same line, E F, at the points E and F respectively.**

*To prove that A C and B D are parallel.*

*Suggestion 1.* *A C* and *B D* either meet or do not meet.

*2.* If they meet, let *O* represent the point of meeting.

*3.* Compare the assumption that they meet at *O* with § 94.

*4.* Do the lines *A C* and *B D* meet?

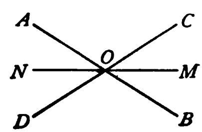
*5.* See § 101.

*Therefore* —

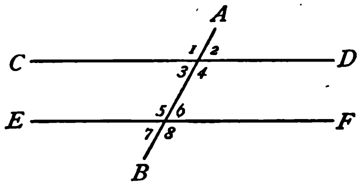
---

Ex. 41. Through two points an inch apart draw two parallel lines. *Sug.* Use corner of card or sheet of paper.

Ex. 42. If two vertical angles are bisected by two straight lines, prove that the bisectors together form one and the same straight line. Prove that *N O M* is a straight line.



**103.** A **transversal** or **secant line** is a line which crosses two or more lines; as the line *A B*.




When a transversal cuts two lines eight angles are formed, *viz.*, the angles 1 to 8 in the figure.

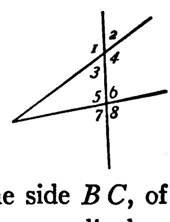
**104.** The **interior angles** are those within, or between, the lines; as 3, 4, 5 and 6.

**105.** The **exterior angles** are those without the lines; as 1, 2, 7 and 8.

**106. Alternate interior angles** are pairs of non-adjacent interior angles on opposite sides of the transversal; as 3 and 6, 5 and 4.

**107. Alternate exterior angles** are pairs of non-adjacent exterior angles on opposite sides of the transversal; as 1 and 8, 2 and 7.

**108. Corresponding angles** are pairs of non-adjacent angles on the same side of the transversal, one exterior and one interior; as 2 and 6,4 and 8, etc.

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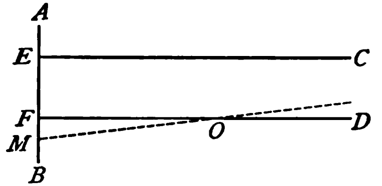
Ex. 43. Draw parallel and transversal lines so as to illustrate all kinds of angles that have been defined.

Ex. 44. If *D* is the middle point of the side *B C*, of triangle *A B C*, and *B E* and *C F* are the perpendiculars from *B* and *C* to *A D*, and *A D* extended, prove that *B E* is equal to *C F*. § 98.

## Proposition XIX.

109. **Theorem.** *If one of two parallel lines is perpendicular to a given line, the other one is perpendicular to the same line.*




**Let E C and F D be two parallel lines, and let E C be perpendicular to A B.**

*To prove that F D is perpendicular to A B.*

*Suggestion 1.* From some point in *F D*, as *O*, draw *O M* to represent a ⊥ to *A B*.

*2.* What relation does *O M* sustain to *E C?* § 102.

*3.* What relation does *F D* sustain to *E C?* Hyp.

*4.* What relation does *M O* sustain to *F D?* *Ax.* 18.

*5.* Then what relation does *F D* sustain to *A B?* Why? Sug. 1.

*Therefore* —

Of what theorem is this the converse?

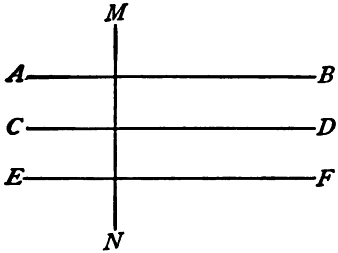
---

Ex. 45. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides, are equal.

Ex. 46. If two lines, *A B* and *C D*, intersect in the point *O*, and if the lines *A C* and *B D* be drawn, *A B* + *C D* is greater than *A C* + *B D*.

## Proposition XX.

110 **Theorem.** *If two lines are parallel to the same line, they are parallel to each other.*




**Let A B and C D each be parallel to E F.**

*To prove A B and C D are parallel.*

*Suggestion 1.* Draw $M N \perp$ to $E F$.

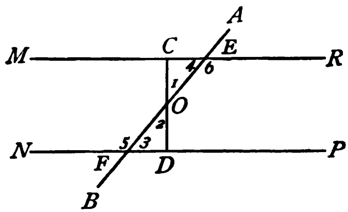
2. How is $M N$ related to $C D$? To $A B$? Why? § 109.

3. How are $A B$ and $C D$ related? Why? § 102.

*Therefore* —

## Proposition XXI.

111. **Theorem.** *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*




**Let M R and N P represent two parallel lines cut by the transversal A B.**

Case I. *To prove that the alternate interior angles 4 and 3 are equal.*

*Suggestion 1.* Through $O$, the middle point of $E F$, draw $C D \perp$ to $N P$.

*2.* What relation does $C D$ sustain to $M R$? Why?

*3.* Compare $\Delta$s $O F D$ and $O E C$. Give auth.

*4.* Then, how do $\angle$s 4 and 3 compare? Why? § 84.

CASE II. *To prove that the alternate interior angles 6 and 5 are equal.*

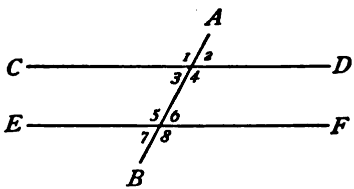
*Suggestion 1.* $\angle 4 + \angle 6 = \angle 5 + \angle 3$. Why?

*2.* Compare $\angle$s 6 and 5.

*Therefore —*

## PROPOSITION XXII.

**112. Theorem.** *If two parallel lines are cut by a transversal, the corresponding angles are equal.*




**Let C D and E F represent two parallel lines, cut by the transversal AB.**

*To prove that the corresponding angles 2 and 6, or 3 and 7, etc., are equal.*

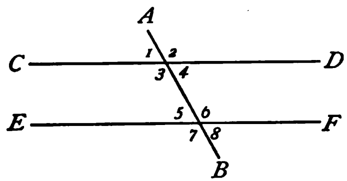
*Suggestion:* See § 111.

---

Ex. 47. In Proposition XIII., place two equal shorter sides upon each other, as $A B$ upon $D E$, connect $C$ and F, and demonstrate the proposition. *Ax. 3.*

## Proposition XXIII.

**113. Theorem.** *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplements of each other.*




**Let C D and E F represent two parallel lines cut by the transversal A B, and let 4 and 6 be two interior angles on the same side of A B.**

*To prove that angles 4 and 6 are supplements of each other.*

*Suggestion 1.* Compare ∠s 3 and 6. Give auth.

*2.* Compare ∠s 3 and 4. § 28.

*3.* How, then, does ∠ 6 compare with ∠ 4? Why?

*Therefore* —

Note.—Several different methods of demonstration should be worked out in Propositions *XXII* and *XXIII.*

---

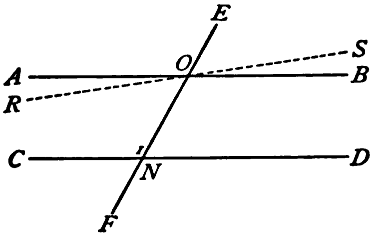
Ex. 48. The line joining the vertices of two isosceles triangles, on opposite sides of the same base, bisects the base and is perpendicular to it.

*Suggestion:* Method used in § 89.

Ex. 49. If a perpendicular is dropped from the vertex of an isosceles triangle to the base, prove (1) that it bisects the base; (2) that it bisects the vertical angle; and (3) that it bisects the triangle.

## Proposition XXIV.

**114. Theorem.** *If two straight lines are cut by the transversal, so that the alternate interior angles are equal, the lines are parallel.*




**Let A B and C D be two straight lines, cut by the transversal E F, so that angles B O F and 1 are equal.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through *O*, draw *R S* to represent a line ∥ to *C D*.

*2.* Compare ∠ *S O F* with ∠ 1. Give auth.

*3.* Compare ∠ *B O F* with ∠ 1. Give auth.

*4.* Compare ∠ *S O F* with ∠ *B O F*. Give auth.

*5.* Since ∠s *S O F* and *B O F* are superimposed and have one side and the vertex common, what relation must *O S* and *O B* sustain to each other? § 17 (e).

*6.* Since *R S* is, by construction, ∥ to *C D*, what relation does *A B* sustain to *C D?* Why?

*Therefore* —

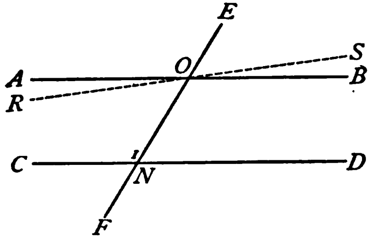
Of what proposition is this the converse?

---

Ex. 50. If two parallel lines are cut by a transversal, prove that the alternate exterior angles are equal.

### Model.

## Proposition XXIV.




**Theorem.** *If two straight lines are cut by a transversal, so that the alternate interior angles are equal, the lines are parallel.*

**Let A B and C D be two straight lines, cut by the transversal E F, so that angles B O F and 1 are equal.**

*To prove that A B and C D are parallel.*

Through $O$, draw $R S$ to represent a line ∥ to $C D$.

The $\angle S O F = \angle 1$. § 111.

$\angle B O F = \angle 1$. Hyp.

$\therefore \angle S O F = \angle B O F$. *Ax.* 1.

$\angle S O F = B O F$ and $O F$ is common, therefore $O S$ and $O B$ coincide. § 17 (e).

But $O S$ is ∥ to $C D$, hence $A B$ is ∥ to $C D$. Q. E. D.

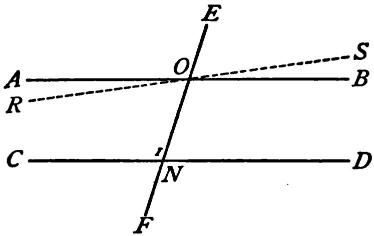
### Review.

If parallel lines are cut by a transversal, what pairs of angles are equal?

How may we know lines are parallel? Name two ways.

## PROPOSITION XXV.

**115. Theorem.** *If two straight lines are cut by a transversal so that the corresponding angles are equal, the two straight lines are parallel.*




**Let A B and C D represent two straight lines cut by the transversal E F so that the corresponding angles E O A and 1 are equal.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through *O*, draw *R S* to represent a line ∥ to *C D*.

*2.* Compare ∠ *E O R* with ∠ 1. Give auth.

*3.* Compare ∠ *E O A* with ∠ 1. Give auth.

*4.* Compare ∠ *E O R* with ∠ *E O A*. Give auth.

*5.* What relation does *O A* sustain to *O R*?

*6.* Since *R S* is, by construction, ∥ to *C D*, what relation does *A B* sustain to *C D*? Why?

*Therefore* —

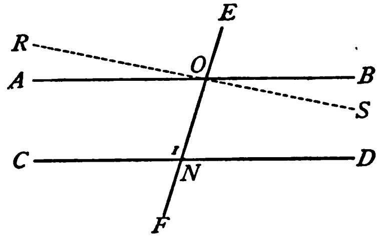
QUERY: If ∠ *1* is a rt. ∠, what previous theorem does *XXV* become?

---

Ex. 51. In the figure for § 112, compare angles 1 and 6; also angles 4 and 7.

## PROPOSITION XXVI.

**116. Theorem.** *If two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are the supplements of each other, the two straight lines are parallel.*




**Let A B and C D represent two straight lines cut by the transversal E F so that angles A O F and E N C are supplements of each other.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Through $O$, draw $RS$ to represent a line ∥ to $CD$.

2. Compare $\angle AOF$ with $\angle ROF$. (See method in § 115.)

Complete the demonstration.

*Therefore* —

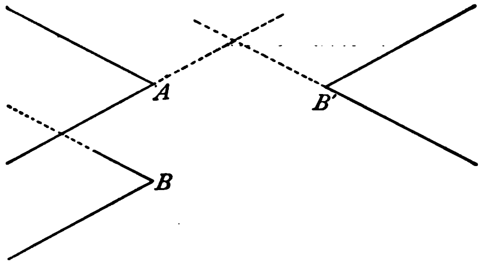
QUERY: If you were to construct a line parallel to a given line, how would you do it?

---

EX. 52. In Fig. for § 103, if angle 1 contains $47\frac{1}{3}$ degrees, how many degrees in angle 4? in angle 2? in angle 8? in angle 6? in angle 5? if angle 3 contains 39 degrees, how many degrees in angle 6? in angle 8? in angle 7? in angle 5? Give authority for each statement.

### Proposition XXVII.

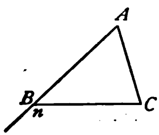
**117. Theorem.** *Two angles which have their sides respectively parallel and extending in the same direction, or in opposite directions, from their vertices are equal.*




**Let A and B represent two angles whose sides are respectively parallel and extend in the same direction from their vertices; and A and B′ two angles whose sides are respectively parallel and extend in opposite directions from their vertices.**

*To prove angles A and B equal, also angles A and B′.*

*Suggestion:* Extend a side of one angle until it meets a side, or an extended side of the other. Complete the demonstration.




**118. Opposite interior angles** of a triangle are the two angles of the triangle not adjacent to the exterior angle of the triangle, as $\angle$s $A$ and $C$. $\angle\, n$ is an exterior $\angle$. § 58.

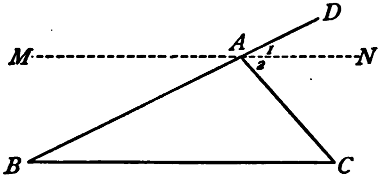
---

Ex. 53. The perimeter of a triangle is less than twice the sum of the medians.

Ex. 54. In what kind of triangle does the bisector of the vertical angle coincide with both the median to the base and the perpendicular from the vertex to the base?

## Proposition XXVIII.

**119. Theorem.** *An exterior angle of a triangle is equal to the sum of the opposite interior angles.*




**Let ABC represent a triangle, DAC an exterior angle, and B and C the opposite interior angles.**

*To prove that angle D A C is equal to the sum of angles B and C.*

*Suggestion 1.* Through the vertex $A$ draw a line $MN$ ∥ to $BC$.

*2.* Compare ∠ 1 with ∠ $B$. Give auth.

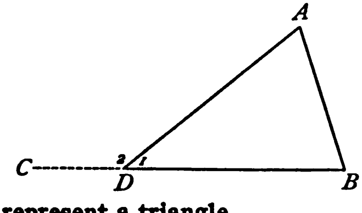
*3.* Compare ∠ 2 with ∠ $C$. Give auth.

*4.* Compare ∠ $DAC$ with ∠ $B$ + ∠ $C$. Give auth.

*Therefore —*

## Proposition XXIX.

**120. Theorem.** *The sum of the interior angles of a triangle is equal to two right angles*




**Let ABD represent a triangle.**

*To prove that the sum of angles A, B, and 1 is equal to two right angles.*

*Suggestion 1.* Extend one of the sides, as *B D*.

*2.* Compare ∠s 1 + 2 with ∠s 1 + *A* + *B*.

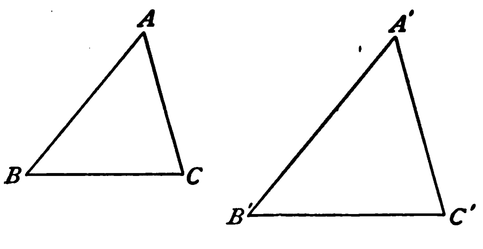
*Therefore* —

**121.** Corollary I. *A triangle can have only one obtuse angle.*

**122.** Corollary II. *Every right triangle has two acute angles, each of which is the complement of the other.*

## Proposition XXX.

**123. Theorem.** *If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.*




**Let ABC and A′B′C′ represent two triangles, having angles A and A′ equal, also angles B and B′.**

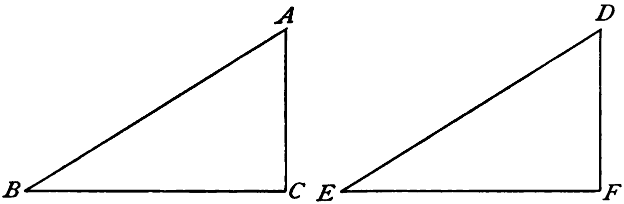
*To prove that angles C and C′ are equal.*

*Suggestion:* See § 120.

**124.** Corollary. *If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.*

## Proposition XXXI.

**125. Theorem.** *If two right triangles have a side and the opposite angle of one equal to the corresponding side and opposite angle of the other respectively, the right triangles are equal in all respects.*



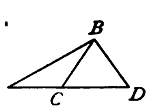

**Let A B C and D E F represent two right triangles, in which A C is equal to D F, and angle B is equal to angle E.**

*To prove that triangles A B C and D E F are equal in all respects.*

*Suggestion:* Compare ∠s *A* and *D*. § 123.
Complete the demonstration.
*Therefore* —

---

Ex. 55. Two triangles having two sides and an angle opposite one of them equal each to each may or may not be equal.

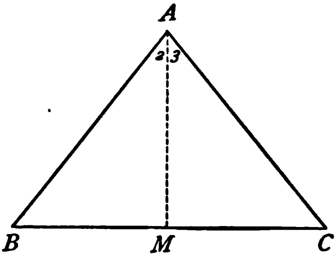



*Suggestion:* If the equal ∠s are acute and opposite the shorter of the two equal sides, the △s may not be equal.

Study the figure and show this to be true. Show the equality of the △s in other cases. *B D* = *B C*.

## Proposition XXXII.

**126. Theorem.** *If two angles of a triangle are equal, the sides opposite them are equal and the triangle is isosceles.*




**Let A B C represent a triangle in which the angle B is equal to angle C.**

*To prove that the side A C is equal to the side A B.*

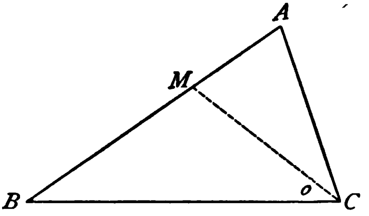
*Suggestion 1.* Drop a ⊥ from *A* to *B C*, as *A M*.
*2.* Compare △s *A M B* and *A M C*. Give auth.
*3.* Compare *A C* with *A B*.
*Therefore* —

---

Ex. 56. If *A B C* is a right triangle with the right angle at *C*, and if through *C* a line meeting the hypotenuse at *D* is drawn in such a manner that angle *A C D* equals angle *A*, prove (1) that *C D* bisects the hypotenuse *A B*, and (2) that if angle *A C D* is not equal to *A*, *C D* does not bisect the hypotenuse.

### PROPOSITION XXXIII.

**127. Theorem.** *If two angles of a triangle are unequal, the sides opposite them are unequal, and the greater side is opposite the greater angle.*




**Let A B C represent a triangle in which the angle C is greater than the angle B.**

*To prove that the side B A is greater than the side C A.*

*Suggestion 1.* Draw *C M* to represent a line making $\angle o$ equal to $\angle B$. Is this possible?

*2.* Compare *B M* and *C M*. Give auth.

*3.* Compare *C M* + *M A* with *C A*, *B A* with *C A*. Give auth.

*Therefore* —

---

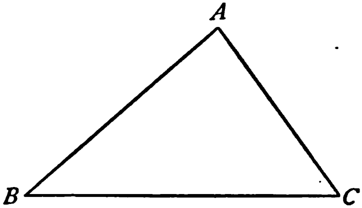
Ex. 57. Prove Proposition XXIX by other methods.

*Suggestion:* Through *A* draw a line ∥ to *D B*. Or extend *D A* and *B A* through *A*, and draw through *A* a line ∥ to *D B*.

Ex. 58. If *A B C* is a right triangle with the right angle at *C*, and if through *C* a line meeting the hypotenuse at *D* is drawn in such a manner that angle *A C D* equals angle *B*, prove that *C D* is perpendicular to the hypotenuse *A B*.

## PROPOSITION XXXIV.

**128. Theorem.** *If two sides of a triangle are unequal the angles opposite them are unequal, and the greater angle is opposite the greater side.*



**Let A B C represent a triangle in which the side A B is greater than the side A C.**

*To prove that angle C is greater than angle B.*

*Suggestion 1.* $\angle C$ equals $\angle B$, is less than $\angle B$, or is greater than $\angle B$.

*2.* If $\angle C = \angle B$, compare $AB$ and $AC$. Can $\angle C = \angle B$? § 126.

*3.* If $\angle$ C is greater than $\angle B$, compare $AB$ and $AC$. § 127.

Can $\angle$ C be greater than $\angle$ B?

*4.* If $\angle C$ is less than $\angle B$, compare $AB$ and $AC$.

Can $\angle C$ be less than $\angle B$?

*Therefore* —

Of what proposition is this the converse?

Compare propositions *IX*, *XXXII*, *XXXIII*, and *XXXIV*.

---

Ex. 59. If from any point in the base of an isosceles triangle perpendiculars are dropped to the sides, these perpendiculars make equal angles with the base.

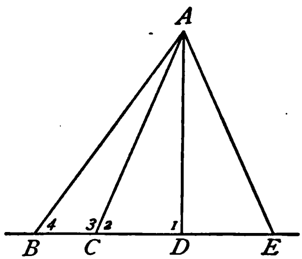
## PROPOSITION XXXV.

**129. Theorem.** *If a perpendicular and oblique lines are drawn from a point to a given line:*

CASE I. *The perpendicular is shorter than any oblique line.*

CASE II. *Two oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.*

CASE III. *Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the more remote is the greater.*




**Let A D be perpendicular to a given line B E, and A B, A C and A E oblique lines, meeting the given line at B, C and E respectively; and let D C be equal to D E and D B greater than D E.**

CASE I. *To prove that A D is shorter than any oblique line from A to B E.*

*Suggestion 1.* How does ∠ 1 compare with ∠ 2? Why?

*2.* How does *A D* compare with *A C*? Why?

*Therefore* —

CASE II. *To prove that A C equals A E.*

*Suggestion 1.* Compare triangles *A D C* and *A D E.*

*2.* Compare *A C* and *A E.* Give auth.

*Therefore* —

CASE III. *To prove that A B is greater than A C or A E.*

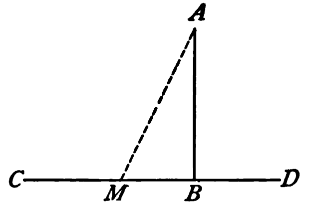
*Suggestion 1.* Compare ∠s 3 and 2; ∠s 3 and 4.

*2.* Compare lines *A B* and *A C.* Give auth.

*Therefore* —

## PROPOSITION XXXVI.

**130. Theorem.** *The shortest line from a point to a straight line is a perpendicular from the point to the line.*




**Let A B represent the shortest line from A to the line C D.**

*To prove that A B is perpendicular to C D.*

*Suggestion 1.* If *A B* is not ⊥ to *C D*, draw *A M* to represent a ⊥ from *A* to *C D.*

*2.* ∴ *A M* is the shortest line from *A* to *C D.* Why? § 129, Case I.

*3.* What relation does *A M* sustain to *C D? Ax.* 16, Hyp.

*4.* What relation does *A B* sustain to *C D?*

*Therefore* —

Of what theorem is this the converse? Compare them.

**131.** The **distance from a point to a line** is the length of the perpendicular from the point to the line.

**132.** Propositions *XXXIV* and *XXXVI* are good illustrations of what is known as the indirect method, or the ***reductio ad absurdum*** method of reasoning. Its peculiarity consists in the fact that the statement of the proposition is not directly proved to be true, but that everything which contradicts the statement of the proposition is shown to lead to some manifest absurdity, and is therefore false. This method often presents difficulties to the beginner on account of the fact that he is obliged to admit, temporarily, and for argument's sake, something which the argument itself goes to destroy. The indirect method is employed in demonstration of the converse of propositions. In applying this method care must be taken that every possible case which contradicts the proposition is considered, and each one shown to lead to an absurdity. Then, and then only, is this method of demonstration rigid.

NOTE.—If the student, in original investigation of the above propositions, or any similar one, should chance to consider the supposition which leads to the truth before one or more of the others, the remaining suppositions should be investigated. When all possible suppositions have been stated, one of which is true, and it has been shown that all but one are false, it is evident that the one remaining must be true.

---

Ex. 60. The line joining the feet of the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides, is parallel to the base.

Ex. 61. The base of an isosceles triangle, together with the bisectors of the angles at the base, forms a second isosceles triangle.

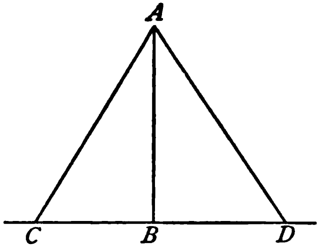
Ex. 62. Two angles having their sides respectively parallel, and each angle, having one side extending in the same direction from their vertices and the other side in opposite directions, are supplements of each other.

Ex. 63. If two triangles have two angles and a side opposite one of them equal respectively to two angles and a corresponding side of the other, the triangles are equal in all respects.

Ex. 64. Two isosceles triangles having equal bases and equal vertical angles are equal.

## Proposition XXXVII.

133. **Theorem.** *Two equal oblique lines, drawn from the same point in a perpendicular to a given line, cut off equal distances from the foot of the perpendicular.*




**Let A B represent a perpendicular to C D, and let A C and A D represent equal oblique lines drawn from A to C D.**

*To prove that B C is equal to B D.*

Of what theorem is this the converse?

*Therefore—*

NOTE.—The pupil should try frequently to demonstrate a proposition with no drawing to assist him. If in recitation he can construct the figure in his mind and make clear his thought to the class, he is worthy of extra credit. Try it with proposition *XXXVII.*

QUERIES —

1. By how many methods can we determine that two lines are parallel? §§ 101, 102, 114, etc.

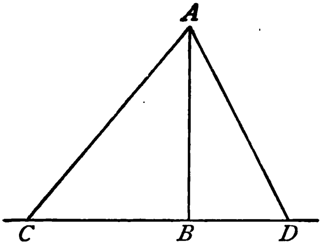
2. By how many methods can we know a line is perpendicular to a line? §§ 23, 109, Ex. 48.

---

Ex. 65. *M* and *N* are two parallel lines. Line *A* is a perpendicular to *M* and line *B* is perpendicular to *N*. Prove that *A* and *B* are parallel.

### PROPOSITION XXXVIII.

**134. Theorem.** *Two unequal oblique lines, drawn from the same point in the perpendicular to a given line, cut off unequal distances from the foot of the perpendicular, the longer line cutting off the greater distance.*




**Let AB represent a perpendicular to CD, and AC and AD unequal oblique lines drawn from A to CD, AC being longer than AD.**

*To prove that BC is greater than BD.*

*Suggestion 1.* $BC = BD$, is less than $BD$, or is greater than $BD$.

2. If $BC = BD$, how would $AC$ compare with $AD$? Why?

3. If $BC$ is less than $BD$, how would $AC$ compare with $AD$? Give auth.

4. How, then, must $BC$ compare with $BD$?

*Therefore* — § 132.

Of what theorem is this the converse?

---

Ex. 66. Each angle of an equilateral triangle is one-third of two right angles, or two-thirds of one right angle.

### Locus of a Point.

**135.** To locate definitely a point in a plane, two conditions limiting its position must be known.

If only one condition is given, the point is to some extent, but not completely, determined. For example, if a point is at a given distance from some fixed point, it is not exactly located, but may move, provided that in its movement it always satisfies the requirement of being at a given distance from the fixed point. The moving point may occupy any position whatever in a line, which will later be defined as the circumference of a circle with the fixed point as the center.

**136.** When the position of a point in a plane is limited to and may be anywhere in a line or group of lines, the line or group of lines is the **locus of the point.**

The locus of a point is both inclusive and exclusive. It must include all the points that satisfy the given condition and exclude all that do not satisfy that condition. For example, if one would know that a circumference or any other line or group of lines is a locus, one must ascertain two things: first, that the point **may be anywhere** in the line or set of lines; second, that the point **must be somewhere** in the line or set of lines; i. e., it cannot be anywhere outside of them.

---

Ex. 67. A straight line drawn from any point in the bisector of an angle to either side and parallel to the other side, makes, with the bisector and the side to which the line is drawn, an isosceles triangle.

## Proposition XXXIX.

**137. Problem.** *To determine the locus of a point at equal distances from the extremities of a given line.*


**Let A B represent a given line.**

*To determine the locus of a point at equal distances from A and B.*

*Suggestion 1.* The problem means that we are to find one or more lines, such that, *first,* any point in them is equally distant from *A* and *B*, and *second,* that no point without them is equally distant from *A* and *B*.

*2.* What line is everywhere equally distant from the extremities of a given line? § 86.

*3.* Compare the distances of any points without the ⊥ from the extremities. § 88.

*4.* What is the required locus?

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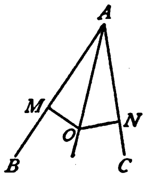



Ex. 68. Prove that every point in the bisector of an angle is equally distant from the sides of the angle.

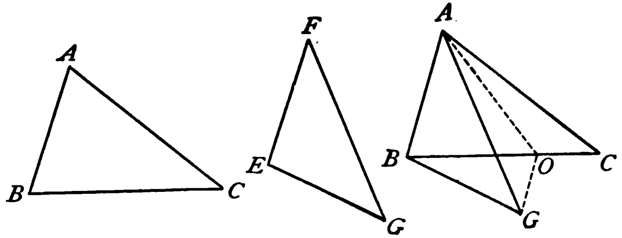
*Suggestion: O M* and *O N* are ⊥ to *A B* and *A C* respectively. § 131.

Prove *O M* equal to *O N*.

Ex. 69. If a line intersects the sides of an isosceles triangle at equal distances from the vertex, the line is parallel to the base.

## PROPOSITION XL.

**138. Theorem.** *If two triangles have two sides of one equal to two sides of the other and the included angles unequal, the remaining sides are unequal, and that side is the greater which is opposite the greater included angle.*




**Let A B C and F E G represent two triangles in which A B is equal to F E, A C is equal to F G, and angle A is greater than angle F.**

*To prove that B C is greater than E G.*

*Suggestion 1.* Place △ *F E G* upon △ *A B C*, so that *F E* coincides with *A B*, *F* upon *A*, and *E* upon *B*.

*2.* Since ∠ *F* is less than ∠ *A*, where does *F G* lie with respect to ∠ *A*? § 17 (e).

*3.* Bisect ∠ *G A C* and extend to meet *B C*, as at *O*. Connect *O* and *G*.

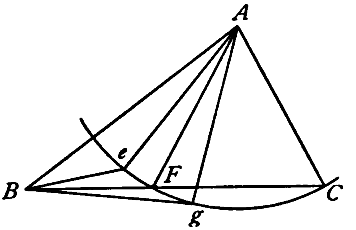
*4.* Compare △s *A O G* and *A O C*.

*5.* Compare lines *O G* and *O C*.

*6.* Compare *B G* with *B O* + *O G*; *B G* with *B C*.

*Therefore* —

In placing $\triangle FEG$ upon $\triangle ABC$, $G$ may be without the $\triangle ABC$, as at $g$, upon line $BC$, as at $F$, or within $\triangle ABC$, as at $e$, as illustrated by the accompanying diagram.




Apply the demonstration above to the figure, with $G$ in each of the three positions.

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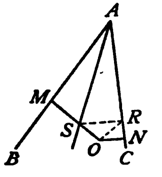
70. Prove that every point not in the bisector of an angle is unequally distant from the sides of the angle.




Let $O$ be at any point not in the bisector of the angle.

Prove that $OM$ and $ON$ are unequal.

*Suggestion 1.* Draw $SR \perp$ to $AN$.

2. Connect $R$ and $O$.

3. Compare $SO + SR$ with $OR$, $OM$ with $OR$, $OM$ with $ON$.

Ex. 71. What is the locus of a point equally distant from the sides of an angle? Exs. 68 and 70.

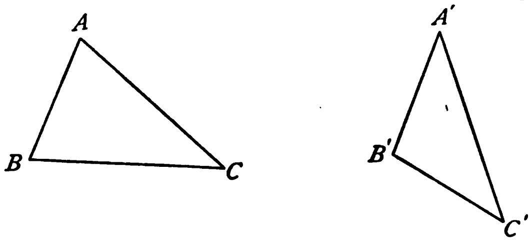
Ex. 72. What is the locus of a point at equal distances from the two intersecting lines? Ex. 71.

Ex. 73. The middle point of the hypotenuse of a right triangle is at equal distances from the three vertices.

*Suggestion:* Ex. 56, or drop $\perp$s from the middle point of the hypotenuse to the legs of the $\triangle$ and compare the triangles formed.

PROPOSITION XLI.

**139. Theorem.** *If two triangles have two sides of one equal to two sides of the other and the third sides unequal, the angles opposite the third sides are unequal, and that angle is the greater which is opposite the greater side.*




**Let A B C and A' B' C' represent two triangles having A B equal to A' B', A C equal to A' C', and B C greater than B' C'.**

*To prove that angle A is greater than angle A'.*

*Suggestion 1.* If $\angle A = \angle A'$, how do $B C$ and $B' C'$ compare? Why?

*2.* If $\angle A < \angle A'$, how do $B C$ and $B' C'$ compare? Why?

*3.* How, then, must $\angle A$ compare with $\angle A'$?

*Therefore* —

**Quadrilaterals.**

140. **Quadrilaterals** are divided into classes, as follows:

(a) The **trapezium,** which has no two sides parallel. Fig. 1.

(b) The **trapezoid,** which has two sides parallel. Fig. 2.

(c) The **parallelogram,** which has its pairs of opposite sides parallel. Fig. 3.

**141.** The parallel sides of a trapezoid are the **bases,** the non-parallel sides are the **legs,** and the line perpendicular to the bases is the **altitude.**

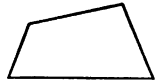


FIG. 1.

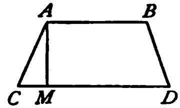



FIG. 2.

In Fig. 2, *A B* and *C D* are the bases, *A C* and *B D* are the legs, and *A M* is the altitude of the trapezoid.

**142.** Any side of a parallelogram may be selected as the base. It is then called the **primary base.** The side opposite the primary base is the **secondary base.** The other sides of the parallelogram are its **sides** or **legs.**

**143.** The **altitude** of a parallelogram is a line perpendicular to the bases, as, *A B* in Fig. 3.

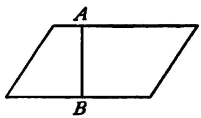



FIG. 3.

NOTE.—Usually the words **lower base** and **upper base** are used instead of primary base and secondary base, respectively, but as Geometry does not take into account the idea, up and down, the terms primary and secondary are preferable.

**144.** An oblique-angled parallelogram is a **rhomboid.** Fig. 3.

**145.** A right-angled parallelogram is a **rectangle.** Fig. 4.

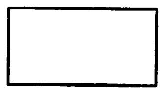


FIG. 4.

**146.** An equilateral rhomboid is a **rhombus.** Fig. 5.

**147.** An equilateral rectangle is a **square.** Fig. 6.

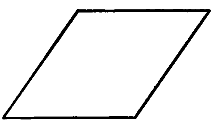


FIG. 5.

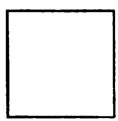
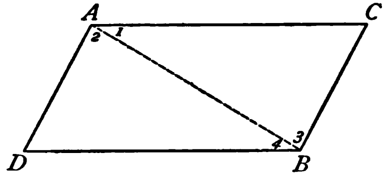


FIG. 6.

The student should make out his own classification of quadrilaterals.

## PROPOSITION XLII.

**148. Theorem.** *The opposite sides and opposite angles of a parallelogram are respectively equal, and adjacent angles are supplements of each other.*




**Let A C B D represent a parallelogram.**

*To prove that A C equals D B; C B equals A D; that angle A and angle B, angle C and angle D are respectively equal; and that the adjacent angles A and C, C and B, B and D, and D and A are respectively supplementary.*

*Suggestion 1.* Draw the diagonal *A B*, and compare △s *A B D* and *A B C*.

2. How do *A C* and *B D* compare? *A D* and *C B?* Complete the demonstration.

*Therefore* —

**149.** COROLLARY 1. *The diagonal of a parallelogram divides it into two equal triangles.*

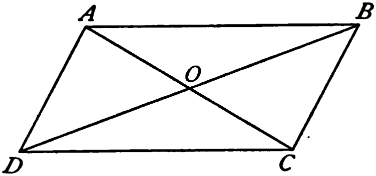
---

Ex. 74. The sum of the exterior angles at the base of any triangle is equal to two right angles plus the vertical angle of the triangle.

Ex. 75. In a triangle *A B C*, the angle *C* is twice the sum of angles *A* and *B*, and angle *B* is twice angle *A;* find all three angles of the triangle.

## Proposition XLIII.

**150. Theorem.** *The diagonals of a parallelogram bisect each other.*




**Let A C represent a parallelogram, and A C and B D the diagonals.**

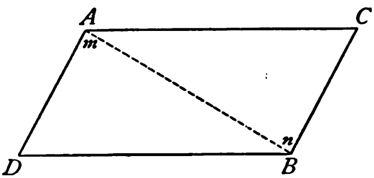
*To prove that A C and D B bisect each other.*

*Suggestion.* Prove by comparison of triangles.

*Therefore* —

## Proposition XLIV.

**151. Theorem.** *If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.*




**Let A C B D represent a quadrilateral in which the side A C is equal and parallel to the side B D.**

*To prove that A C B D is a parallelogram.*

*Suggestion 1.* What is the definition of a ▱?

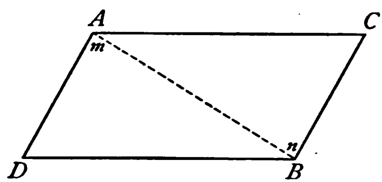
*2.* How much of the definition is given in the Hyp.? What remains to be proved?

*3.* Draw the diagonal *A B*. Compare △s *A B C* and *A B D*. Compare ∠ *m* and ∠ *n*. Complete the demonstration.

*Therefore* —

PROPOSITION XLV.

**152. Theorem.** *A quadrilateral whose opposite sides are equal is a parallelogram.*




**Let C D represent a quadrilateral in which A C is equal to D B, and A D is equal to C B.**

*To prove that A C B D is a parallelogram.*

*Suggestion 1.* Draw the diagonal *A B*.

*2.* Two definitions of a ▱ may now be used. §§140 (c), 151.

*3.* How much of either is given in the theorem? What remains to be proved?

Work out a demonstration by use of each definition.

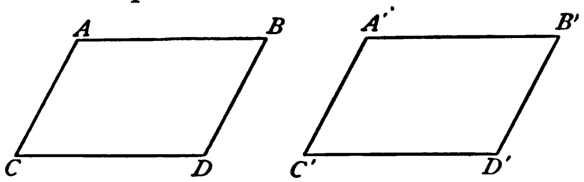
State the different definitions of a parallelogram that have now been established. Commit each to memory.

NOTE.—The pupil should carefully distinguish between a **property** of a figure and a **definition**.

A definition of a figure is such a description of it as serves to distinguish it from every other figure (§ 151). A property of a figure is a statement of it which is true, and may also be true of other figures (§ 148).

## PROPOSITION XLVI.

**153. Theorem.** *Two parallelograms which have two sides and the included angle of one equal to two sides and the included angle of the other, each to each, are equal in all respects.*




**Let AD and A′D′ represent two parallelograms in which AB is equal to A′B′, AC is equal to A′C′ and angle A is equal to angle A′.**

*To prove that A D and A′ D′ are equal in all respects.*

*Suggestion 1.* Place ▱ $A\,D$ upon ▱ $A'\,D'$ so that $A\,B$ coincides with $A'\,B'$, $A$ upon $A'$, and $B$ upon $B'$, and so that the two figures fall upon the same side of $A'\,B'$.

2. What direction does $A\,C$ take? Why?

3. Where does the point $C$ fall? Why?

4. What direction does $C\,D$ take? Why? *Ax.* 18.

5. Where does the point $D$ fall? Why?

Complete the demonstration.

*Therefore* —

---

Ex. 76. If a parallelogram has one right angle it is a rectangle.

Ex. 77. A quadrilateral is a parallelogram if its diagonals bisect each other.

Ex. 78. A quadrilateral is a parallelogram if its opposite angles are equal.

## PROPOSITION XLVII.

154. **Theorem.** *The diagonals of a square or rhombus bisect the angles of the square or rhombus.*

**Let A B C D represent a square or rhombus, A C and B D the diagonals.**

*To prove that angles A, B, C,* and *D* are bisected.

*Suggestion.* Try to demonstrate this proposition by constructing the figure in your mind, and thus establish the theorem without the aid of a drawing.

---

Ex. 79. The diagonals of a square and rhombus intersect at right angles.

Try to find at least three methods by which to prove this exercise.

Ex. 80. Given a square *A D B C*. Draw diagonal *C D*.

Lay side *C B* upon the diagonal, *C* upon *C*, and *B* upon some point, as *E*. At *E* erect a peroendicular and extend to the side *D B*, meeting it at *F*.

Prove that *D E* is equal to *E F* and cqual to *F B*.

*Suggestion.* Connect *C* and *F*.

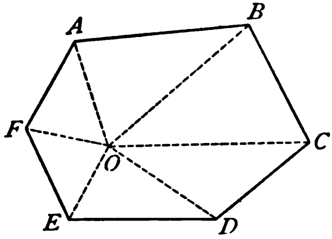
Ex. 81. If the diagonals of a parallelogram intersect at right angles, the figure is a square or rhombus.

Try to prove by at least two methods.

Ex. 82. If a straight line intercepted between parallel lines is bisected, any other straight line drawn through the point of bisection to meet the parallel lines is also bisected at that point. The two intersecting lines intercept equal portions on the parallel lines.

## Proposition XLVIII.

**155. Theorem.** *The sum of the interior angles of any convex polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.*




**Let A D represent any convex polygon.**

*To prove that the sum of the interior angles of the polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.*

*Suggestion 1.* Connect each vertex with $O$, any point within the polygon.

*2.* If the polygon has $n$ sides, how many $\Delta$s are formed?

*3.* How many right $\angle$s in the sum of the interior angles of all of the $\Delta$s?

*4.* What is the sum of the $\angle$s about $O$?

*5.* Then, how many right $\angle$s in the sum of the interior $\angle$s of the polygon?

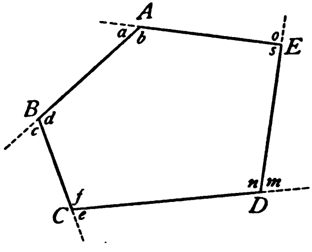
*Therefore* —

---

Ex. 83. Draw all possible diagonals of a polygon from a given vertex, and prove Prop. *XLVIII* by another method.

## PROPOSITION XLIX.

**156. Theorem.** *The sum of the exterior angles of any convex polygon, formed by extending each side through one vertex in order, is equal to four right angles.*




**Let A E D, etc., represent a convex polygon.**

*To prove that the sum of the exterior angles a, c, e, etc., is equal to four right angles.*

*Suggestion 1.* What is the sum of $\angle$s $a$ and $b$? Of $\angle$s $c$ and $d$? Of $\angle$s $f$ and $e$, etc.?

*2.* If the polygon has $n$ sides, the sum of all the exterior and interior $\angle$s equals how many right $\angle$s?

*3.* How many right $\angle$s in $\angle a + \angle c + \angle e$, etc. *Therefore* —

---

Ex. 84. How many right angles in the sum of the interior angles of a pentagon? of a triangle? of a hexagon? of an octagon? of a decagon, etc.?

Ex. 85. How many right angles in the interior angles of a concave polygon? Compare with § 155.

## Review.

**157.** Name all the authorities:

1. (*a*) by which two triangles may be determined equal.

(*b*) by which two angles may be determined equal.

(*c*) by which two lines may be determined parallel?

(*d*) by which a line may be known to be perpendicular to another line.

(*e*) by which a quadrilateral may be known to be a parallelogram? a parallelogram to be a rectangle? a parallelogram to be a square or rhombus?

2. Name all the loci thus far demonstrated.

3. How may triangles be known to be isosceles?

4. State the relation that exists between angles of a triangle and sides opposite; between sides and angles opposite.

---

Ex. 86. What is the locus of a point two inches from a given line indefinite in length? Three inches from the line?

Ex. 87. Find a point $x$ that is four inches from a given line and also in another given line.

Use locus in determining $x$. Is it possible to find more than one $x$? Suppose the two lines are parallel and four inches apart; parallel and any other distance apart?

Ex. 88. Find point $x$ if it is in a given line and also equally distant from two intersecting lines. Ex. 72.

Ex. 89. Find point $x$ if it is equally distant from two intersecting lines and also a given distance from a given line. Ex. 86.

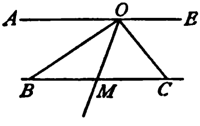
Ex. 90. Find point $x$ if it is equally distant from two intersecting lines, and also a given distance from a given line.

Ex. 91. Find point $x$ if it is equally distant from the extremities of a given line, and also a given distance from a given line.

Ex. 92. The three perpendicular bisectors of the sides of a triangle meet at a point.

*Sug.:* Use locus in demonstrating Exs. 90, 91, and 92.

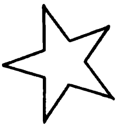
Ex. 93. In the figure, $AE$ and $BC$ are parallel, and $MO$ is a transversal. $BO$ bisects the angle $AOM$, and $CO$ bisects the angle $EOM$. Prove that $BM$ is equal to $MC$.




Ex. 94. If the legs of a trapezoid are equal, they make equal angles with the bases. *Sug.:* Draw through a vertex of the shorter base a line parallel to one of the legs.

Ex. 95. If the legs of a trapezoid make equal angles with the bases the legs are equal.

Ex. 96. The sum of the angles at the vertices of a five-pointed star is equal to two right angles.



Ex. 97. $ABC$ is an isosceles triangle, $D$ any point in the base $BC$, $DM$ a line parallel to $AB$, and $DN$ a line parallel to $AC$, $M$ and $N$ being points in $AC$ and $AB$ respectively. Prove that the perimeter of $AMDN$ is equal to the legs of the triangle.

Ex. 98. A line that embraces the vertex of an angle and a point equidistant from the sides, bisects the angle.

Ex. 99. The difference between any two sides of a triangle is less than the third side.

# CHAPTER II.

## THE CIRCLE.

### Definitions.

**158.** A **circle** is a portion of a plane bounded by a curved line, all points of which are equally distant from a fixed point within.

**159.** The fixed point is the **center** of the circle, and the bounding line is the **circumference** of the circle.

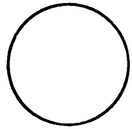


FIG. 1.

NOTE—In higher branches of mathematics the word circle is also used to denote what is here defined as the circumference; that is, the curved *line* bounding a portion of a plane instead of that portion of a plane itself, but in this book the above definitions will be adhered to.

**160.** A **radius** is any straight line drawn from the center of a circle to its circumference, as $A\ O$, Fig. 2.

**161.** A **diameter** is any straight line drawn through the center, terminated by the circumference, as $C\ B$, Fig. 2.

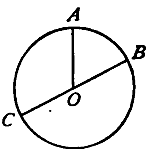



FIG. 2.

**162.** COROLLARY.—*From the definition of a circle, all radii of the same circle are equal; also, all diameters of the same circle are equal, each diameter being twice the radius.*

**163.** An **arc** of a circle is any portion of its circumference, as line $A\ M\ B$, Fig. 3.

**164.** A **chord** of a circle is a straight line joining any two points of the circumference, as $A\ B$, Fig. 3.

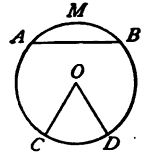



FIG. 3.

(*a*) When the extremities of a chord and an arc are the same, the chord **subtends** the arc.

(*b*) In Fig. 3, the chord *A B* subtends the arc *A M B*, and also the arc *A C D B*. Thus, any chord subtends two arcs, which, together, make up the whole circumference. When an arc and its chord are spoken of, the smaller of the two arcs is always understood, unless the other is specifically stated. The terms **major** and **minor arcs** are sometimes used.

**165.** A **secant** of a circle is any straight line meeting the circumference in two points, and passing through the circumference in at least one of them, as *C D E*, Fig. 4.

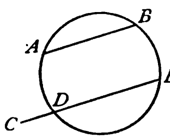


FIG. 4.

A secant is a chord extended.

**166.** A **tangent** to a circle is a line which touches the circumference at one, and only one, point.

In Fig. 5, the line *M N* is tangent to the circle, and the circle is tangent to the line.

(*a*) The line may be straight or curved. If the curved line is a circumference, a circle is tangent to another circle.

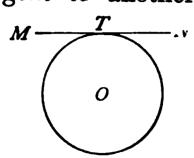


FIG. 5.

(*b*) The point common to the line and the circle or to the two circles is the **point of contact** or **point of tangency**.

**167.** A **segment** of a circle is a portion of a circle bounded by an arc and its subtending chord, as segment *A M B*, Fig. 3.

**168.** A **sector** of a circle is a portion of a circle bounded by two radii and the intercepted arc. In Fig. 3, the sector *C O D* is bounded by the arc *C D*, and the radii *O C* and *O D*.

**169.** The arc intercepted between two radii **subtends** the angle made by the radii; arc *C D*, Fig. 3, subtends angle *C O D*.

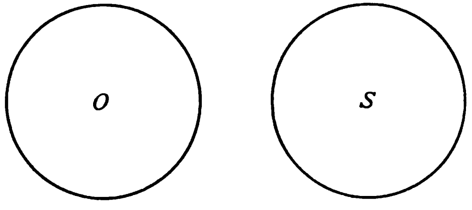
**170.** An **angle at the center** of a circle is an angle formed by two radii.

**171.** **Concentric circles** are circles that have the same center.

**172.** A circle may be **read** by naming the letter at the center of the circle, or by naming two or more letters on the circumference.

### Proposition I.

**173. Theorem.** *Two circles are equal if the radius of one equals the radius of the other.*




**Let O and S represent two circles having equal radii.**

*To prove that circle O and circle S are equal.*

*Suggestion* 1. Place ⊙ *O* upon ⊙ *S*, with the center *O* upon the center *S*.

2. Where must the circumference of *O* fall with respect to the circumference of *S*? Why?

*Therefore* —

### INDIRECT METHOD.

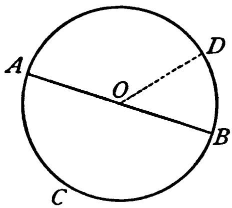
*Suggestion 1.* Same as above.

*2.* Suppose some part of the circumference of one ⊙ should fall outside of the circumference of the other ⊙, how would the radii compare?

*Therefore*—

### PROPOSITION II.

**174. Theorem.** *A diameter divides a circle into two equal parts.*




**Let A D B C represent a circle, and A B a diameter.**

*To prove that A B divides the circle into two equal parts.*

*Suggestion 1.* Let *D* be any point in the arc *A D B*, except *A* and *B*. Connect the center *O* with *D*.

*2.* Revolve the segment *A D B* upon the line *A O B*, as an axis, into the plane *A C B*.

*3.* Where does the point *D* fall? § 158.

*4.* Since *D* is any point in the arc *A D B*, compare the segments *A D B* and *A C B*. *Ax.* 13.

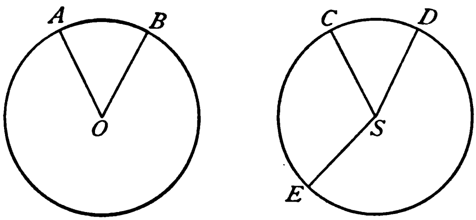
*Therefore*—

**175.** COROLLARY.—*A diameter divides the circumference into two equal parts.*

**176.** A **semicircle** is a segment of a circle bounded by a diameter and the arc it subtends, as *A D B*, Fig., § 174.

## Proposition III.

**177. Theorem.** *In the same circle, or in equal circles, equal angles at the center intercept equal arcs at the circumference, and of two unequal angles, the greater intercepts the greater arc.*




**Let O and S represent equal circles, and let A O B and C S D represent equal angles at the centers O and S respectively.**

Case I. *To prove that arc A B is equal to arc C D.*

*Suggestion 1.* Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *A O* upon *C S*. Where does the circumference of ⊙ *O* fall? Why?
Where does point *A* fall? Why?

*2.* What direction does *O B* take? Why?

*3.* What is the location of arc *A B*? Why?

*Therefore* —

**Let angle C S E be greater than angle A O B.**

Case II. *To prove that arc E C is greater than arc A B.*

*Suggestion 1.* Place ⊙ *O* upon ⊙ *S*, the center *O* upon the center *S*, and *O B* upon *S C*. Where does *O A* fall with respect to ∠*C S E*? Why? § 17 (e).

2. Where does point $A$ fall with respect to arc $C E$? Why? Compare arc $A B$ and $C E$.

*Therefore —*

## Proposition IV.

**178. Theorem.** Conversely. *In the same circle, or in equal circles, equal arcs subtend equal angles at the center, and of two unequal arcs, the greater arc subtends the greater angle at the center.*

**Let O and S (§ 177) represent equal circles, and A B and C D equal arcs.**

Case I. *To prove that angle $O$ is equal to angle $C S D$.*

*Suggestion.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, so that arcs $A B$ and $C D$ coincide. Why is this possible?

Complete the demonstration.

*Therefore —*

**Let arc E C be greater than arc A B.**

Case II. *To prove angle $C S E$ is greater than angle $O$.*

*Suggestion 1.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, and $B$ upon $C$. Where does point $A$ fall with respect to arc $C E$? Why?

2. Where does line $O A$ fall? Why?

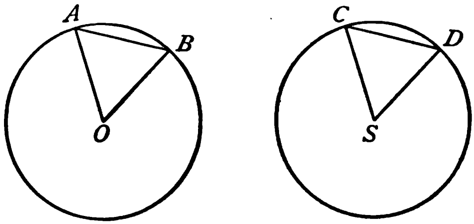
*Therefore —*

---

Ex. 100. The locus of a point at a given distance from a given point is the circumference of a circle, drawn with the given point as a center and with the length of the given distance as a radius. § 136.

## Proposition V.

179. **Theorem.** *In the same circle, or in equal circles, chords which subtend equal arcs are equal.*




**Let O and S represent equal circles, in which arc A D is equal to arc C D.**

*To prove that chord A B is equal to chord C D.*

*Suggestion 1.* Draw the radii $O A$, $O B$, $S C$, and $S D$.

*2.* Compare $\angle$s $O$ and $S$. Give auth.

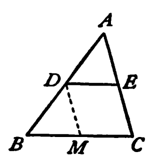
*3.* Compare $\triangle$s $A O B$ and $C S D$. Give auth.

*4.* Compare chord $A B$ with chord $C D$.

*Therefore* —

---

Ex. 101. If a line, parallel to the base of a triangle, bisects one side, it bisects the other side also, and is equal to one-half of the base. $A B$ is bisected at D, and $D E$ is parallel to $B C$.




Prove $A E = E C$, and $D E = \frac{1}{2} B C$.

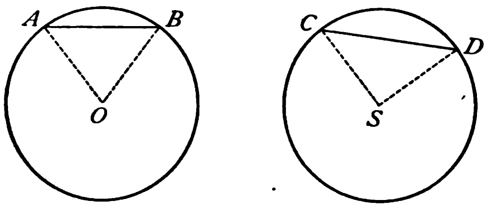
*Suggestion.* Draw $D M \parallel$ to $A C$.

Ex. 102. With the same construction as in exercise 101, connect $E$ and $M$, and prove the exercise.

Ex. 103. If a line bisects the two legs of a triangle, prove that it is parallel to the base, and equals one-half of the base. Direct and indirect methods.

### Proposition VI.

180. **Theorem.** *In the same circle, or in equal circles, two chords which subtend unequal arcs are unequal, that chord being greater which subtends the greater arc.*




**Let O and S represent equal circles, and let the arc A B be less than the arc C D.**

*To prove that the chord A B is less than the chord C D.*

*Suggestion 1.* Connect the extremities of the arcs with their respective centers.

*2.* Compare $\angle S$ with $\angle O$. § 178.

*3.* Compare chord $CD$ with chord $AB$. § 138.

*Therefore* —

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Ex. 104. A diameter is greater than any other chord.

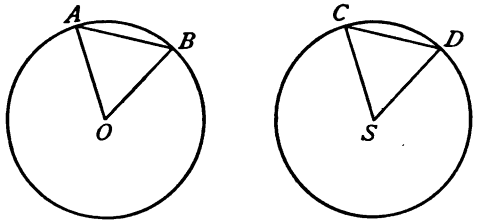
*Suggestion.* Draw any chord, not a diameter, and draw radii to the extremities. Compare the chord with the sum of the radii.

Ex. 105. If the diagonals of a parallelogram are equal, prove that the parallelogram is a rectangle.

Ex. 106. Given two parallel lines and a transversal. If each of the two interior angles on the same side of the transversal is bisected, prove that the bisectors are perpendicular to each other.

## PROPOSITION VII.

**181. Theorem.** CONVERSE OF PROPOSITION V. *In the same circle, or in equal circles, arcs which are subtended by equal chords are equal.*




**Let O and S represent equal circles in which the chord A B equals the chord C D.**

*To prove that the arc A B is equal to the arc C D.*

*Suggestion 1.* Draw radii *O A*, *O B*, *S C*, and *S D*.

*2.* Compare △s *A O B* and *C S D*. Give auth.

*3.* Compare ∠s *O* and *S*. Give auth.

*4.* Complete the demonstration.

*Therefore* —

---

182. A **diameter** of a quadrilateral is a line which joins the middle points of two opposite sides.

Ex. 107. The diameter of a parallelogram divides it into two equal parallelograms.

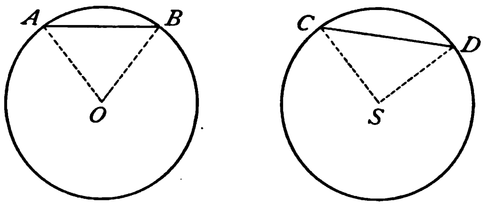
Ex. 108. The two diameters of a parallelogram bisect each other.

Ex. 109. If two adjacent sides of a rectangle are equal the figure is a square.

Ex. 110. Two lines perpendicular, respectively, to each of two intersecting lines, cannot be parallel.

### PROPOSITION VIII.

**183. Theorem.** CONVERSE OF PROPOSITION VI. *In the same circle, or in equal circles, two arcs which are subtended by unequal chords are unequal, and that arc is the greater which is subtended by the greater chord.*




**Let O and S represent equal circles, and let the chord A B be less than the chord C D.**

*To prove that the arc A B is less than the arc C D.*

*Suggestion 1.* Connect the extremities of the chords with the respective centers. Compare ∠ $O$ with ∠ $S$ in △s $A\,O\,B$ and $C\,S\,D$. § 139.

*2.* Compare arc $A\,B$ with arc $C\,D$. Give auth.

*Therefore* —

---

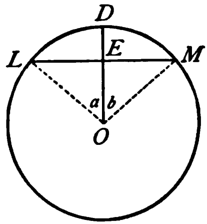
Ex. 111. If a circumference is divided into four equal parts by four equal chords, the resulting quadrilateral is a square.

Ex. 112. If one of the equal sides of an isosceles triangle is extended through the vertex, and the exterior angle formed is bisected, the bisector is parallel to the base.

Ex. 113. In § 153 draw the diagonals $C\,B$ and $C'\,B'$, and prove the proposition by comparison of triangles.

### Proposition IX.

184. **Theorem.** *A radius which is perpendicular to a chord of a circle, bisects the chord and its subtended arc.*




**Let O D represent a radius perpendicular to the chord L M.**

*To prove that O D bisects the chord L M and arc L M.*

*Suggestion 1.* Let *E* be the point of intersection of *O D* and *L M*.

2. Compare △s *L E O* and *M E O*.

Complete the demonstration. § 177.

*Therefore —*

### Proposition X.

185. **Theorem.** *A radius which bisects a chord of a circle is perpendicular to the chord.*

**Let the radius O D, in figure § 184, bisect the chord L M.**

*To prove O D is perpendicular to L M.*

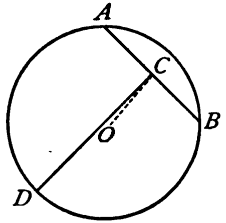
Query: How could this proposition be used to bisect an arc?

---

Ex. 114. A line which bisects a chord and its subtended arc cuts the center of the circle.

## Proposition XI.

**186. Theorem.** *A line perpendicular to a chord at its middle point, passes through the center of the circle.*




**Let A B represent a chord, and C D a perpendicular to A B at its middle point C.**

*To prove that C D passes through the center of the circle.*

*Suggestion 1.* From *O*, the center of the circle, drop a ⊥ to the chord *A B*. § 184.

2. Where does *C O* lie with respect to *C D?* Why?

ANOTHER METHOD. *Suggestion 1.* What is the locus of a point equally distant from *A* and *B?*

2. Locate the center *O* with respect to this locus. Give auth.

*Therefore —*

187. COROLLARY.—*A line perpendicular to a chord at its middle point, bisects the subtended arc.*

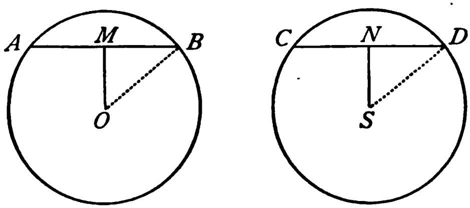
---

Ex. 115. If the line joining the middle points of two chords passes through the center of the circle, the chords are parallel.

Ex. 116. A radius drawn to the middle point of an arc bisects its subtending chord.

## PROPOSITION XII.

**188. Theorem.** *In the same circles or in equal circles, equal chords are equally distant from the center, and of two unequal chords, the greater is nearer the center.*




**Let AB and CD represent equal chords in the equal circles O and S, and OM and SN their respective distances from the centers O and S.**

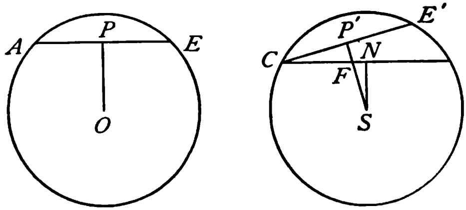
CASE I. *To prove that OM is equal to SN.*

*Suggestion 1.* Draw the radii *OB* and *SD*.

What relation must *MO* and *NS* bear to *AB* and *CD*, respectively? § 130.

*2.* Compare △ *OMB* with △ *SND*.

*3.* Compare *OM* and *SN*.




**Let chord AE be less than chord CD, and let OP and SN represent the distances of the chords from their centers.**

CASE II. *To prove that O P is greater than N S.*

*Suggestion 1.* Compare arc $A E$ and $C D$.

*2.* Place ⊙ $O$ upon ⊙ $S$, the center $O$ upon the center $S$, and point $A$ upon point $C$.

Where does point $E$ fall with respect to point $D$? Sug. 1.

Where does line $A E$ fall with respect to segment $C E' D$?

*3.* Compare $N S$ and $F S$; $N S$ and $P' S$; $N S$ and $O P$.

*Therefore* —

QUERY: Why does $P' S$ cross the line $C D$? Sug. 2.

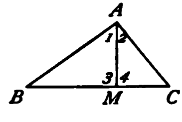
COROLLARY.—*In the same circle or in equal circles, two chords equally distant from the center are equal, and of two chords unequally distant from the center, that is the greater which is nearer the center.*

---

Ex. 117. Through a given point within a circle, the smallest possible chord is the one that is perpendicular to the radius passing through the point.

Ex. 118. $A B C$ is an isosceles triangle whose base is $B C$ and whose vertex is $A$. Extend $B A$ through $A$ to $O$, making $A O$ equal to $B A$. Connect $O$ and $C$. Prove that $O C$ is perpendicular to $B C$. (Ex. 112.)

Ex. 119. If a perpendicular is drawn from the vertex of the right angle of a right-angled triangle to the hypotenuse, prove that the two triangles formed are mutually equiangular.

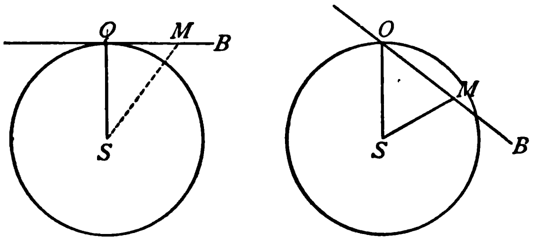



Ex. 120. Prove in Ex. 119 that each segment of the triangle is mutually equiangular to the whole triangle.

Ex. 121. What is the locus of a point equally distant from two parallel straight lines.

## Proposition XIII.

**189. Theorem.** *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*




**Let O B represent a straight line perpendicular to the radius O S at its extremity O.**

*To prove that O B is tangent to the circle.*

*Suggestion 1.* What must be proved to know that *O B* is a tangent? § 166.

2 How much of what must be known is given in the theorem?

3. What remains to be proved?

4. Let *M* represent any point except *O* in line *O B*. Draw *S M*.

5. Compare *O S* and *S M* in respect to length.

6. Where is point *M* in respect to the circle?

*Therefore* —

Query: How could this proposition be used to draw a tangent to a circle?

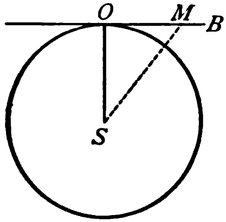
---

Ex. 122. The bisectors of the interior angles of a parallelogram form a rectangle. Suppose the parallelogram is equilateral?

## PROPOSITION XIV.

**190. Theorem.** CONVERSE OF PROPOSITION XIII.

*If a straight line is tangent to a circle, the radius meeting it at the point of tangency is perpendicular to it.*




**Let OB represent a tangent to the circle S, O the point of tangency, and SO the radius drawn to the point of tangency.**

*To prove that S O is perpendicular to O B.*

*Suggestion 1.* Draw *S M*, a line from *S* to any point except *O* in *O B*.

*2.* Compare *S O* and *S M* in respect to length.

*3.* § 130.

*Therefore* —

**191.** COROLLARY 1.—*At any point in a circumference, one, and only one, tangent can be drawn.*

**192.** COROLLARY 2.—*A straight line perpendicular to a tangent at the point of tangency cuts the center of the circle.*

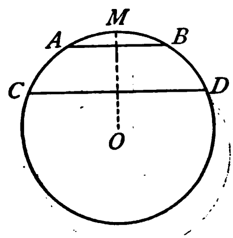
---

Ex. 123. All chords of a circle that are tangent to a concentric circle are equal.

Ex. 124. If two tangents are drawn to a circle at the ends of a diameter, they are parallel.

## Proposition XV.

**193. Theorem.** *Arcs of a circle intercepted by parallel lines are equal.*




**Let A B and C D represent two parallel lines intercepting the arcs A C and B D.**

*To prove that arc A C is equal to arc B D.*

*Suggestion 1.* Drop a ⊥ from *O* to *C D*, and extend it to meet the circumference, as at *M*.

*2.* How is *O M* related to *A B*?

*3.* Complete the demonstration. § 184.

*Therefore* —

### Review.

**194.** State all the possible cases in which two parallel lines may intercept two arcs, and prove them equal in each case.

---

Ex. 125. Two tangents drawn to a circle from the same point are equal. *Suggestion.* Connect the center with the given point. § 190.

Ex. 126. If a tangent and a chord are parallel, prove that they intercept equal arcs.

## PROPOSITION XVI.

**195. Theorem.** *Through three points, not in the same straight line, one circumference, and only one, is possible.*


**Let A B C represent three given points not in the same straight line.**

*To prove that through A, B, and C one circumference, and but one, is possible.*

*Suggestion 1.* What is the locus of a point equally distant from $A$ and $C$? Why is this line a locus of the centers of circles whose circumferences cut $A$ and $C$?

*2.* What is the locus of a point equally distant from $A$ and $B$? Give auth. Of what centers is this line a locus?

*3.* Will these two loci intersect? Why? Ex. 110.

*4.* What relation does this point bear to $A$, $B$, and $C$? Why?

*5.* Can a circumference pass through the points $A$, $B$, and $C$? Why?

*6.* Can there be more than one such circumference? Why? § 136.

*Therefore —*

QUERY: How could you find the center of a circle if it were not known?

## MEASUREMENT.

**196. Quantity** is any magnitude which can be measured by a unit of the same kind.

**197.** The quantities used in geometry are the geometric magnitudes, as, lines, surfaces, and solids.

**198.** To **measure a quantity** is to find out how many times it contains another selected quantity of the same kind, called the unit of measure.

**199.** In everyday experience, the **unit of measure** is a standard accepted by general consent; as a foot, a square yard, a ton, a cord, etc.

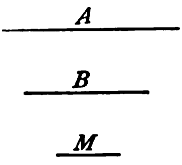
**200.** The **numerical measure of a quantity** is the number which expresses how many times the unit of measure is contained in the given quantity.

**201.** A careful distinction should be made between quantity, and number which is the measure of quantity. These terms are sometimes carelessly confused. 26 gallons, 3 feet, 5 pints, and 29 square feet are quantities; 26, 3, 5, and 29 are numbers. The units of measure by which we derive these numbers from the quantities are, respectively, a gallon, a foot, a pint, and a square foot.

**202.** The **ratio of one quantity to another** is the number of times the first contains the second.

(*a*) In other words the ratio of one quantity to another is the numerical measure of the first regarding the second as the unit of measure; or, having measured both quantities by the same unit, the quotient of the numerical measure of the first divided by the numerical measure of the second.

To illustrate, the ratio of line $A$ to line $B$ is the number of times $A$ contains $B$, regarding $B$ as the unit of measure. The result may be obtained by laying $B$ off upon $A$ as many times as possible; or, $A$ and $B$ may be measured by the same unit of measure, $M$, in which case the ratio of $A$ to $B$ is the number of times the numerical measure of $A$ contains the numerical measure of $B$. Suppose $M$ is contained $c$ times in $A$, and $d$ times in $B$, then the ratio of $A$ to $B$ is equal to $c$ divided by $d$, as $c$ and $d$ are the numerical measures of $A$ and $B$, respectively.




(1) As ratio plays an important part in subsequent geometry, it is important that the pupil have clear and definite ideas of what a ratio actually is, and the relation it sustains to the subject of division.

From the definition of the ratio of two quantities, as the number of times the first contains the second, it follows that a ratio can exist only between quantities of the same kind, and also that the ratio of two quantities is always a pure number. For example, 6 feet contains 3 feet twice; hence the ratio of 6 feet to 3 feet is 2.

Division has been defined as the process of finding how many times one number or quantity is contained in another number or quantity of the same kind; or, the process of separating a number or quantity into a given number of equal parts and taking one of them. In the first conception of division in the definition, the division of measurement, the quotient is identical with the ratio; the divisor and dividend, if quantities, must have a like unit of measure, making the quotient a pure number, or the division is unthinkable. 6 feet ÷ 3 feet = 2; or, $\frac{6 \text{ ft.}}{3 \text{ ft.}} = \frac{6}{3} = 2$. This idea may be expressed as 6 feet = 2 times 3 feet.

(2) In the second kind of division, the division of separation or partition, the divisor is a pure number and the dividend and quotient may be numbers, or like quantities. 16 gallons ÷ 2 means that 16 gallons are to be divided into two equal parts, and one of the parts taken. In common phrase, it means to find ½ of 16 gallons. As ratio identifies itself only with the division of measurement, we shall be concerned with the principles underlying only that kind of division.

(3) Since ratio in the following discussion is based upon the alge-

braic conception of division, and not upon the Euclidean definition, which is much too difficult for beginners, the division or fractional form of statement is most expressive. Hence, the ratio of 6 feet to 2 feet is best expressed as $\frac{6 \text{ ft.}}{2 \text{ ft.}}$. The statement referred to, § 202 (a), that the ratio of quantities $A$ and $B$ equals the ratio of the numbers $c$ and $d$, is best expressed as $\frac{A}{B} = \frac{c}{d}$. This form will be invariably used in the following pages.

**203.** Two quantities of the same kind are **commensurable** when each contains the same unit of measure an integral number of times.

(*a*) Arithmetic is particularly concerned with commensurable quantities.

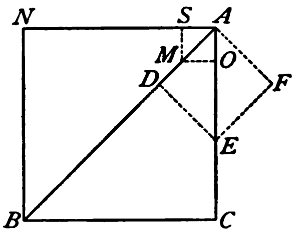
**204.** In determining the common unit of measure of two commensurable quantities, the arithmetical method for finding the greatest common divisor may be followed, which is, to divide the greater quantity by the less, to divide the divisor by the remainder, this remainder by the second remainder, etc., until the division is found to be exact. The exact divisor is the greatest common unit of measure of the two quantities.

To find the greatest common unit of two lines, or other magnitudes, by the arithmetical method, is a valuable geometrical exercise and useful review of arithmetic.

**205.** Two quantities of the same kind are **incommensurable** when they cannot be exactly divided by the same unit of measure, however small, an integral number of times.

(*a*) This idea is difficult of conception. To illustrate: The diagonal and side of a square may be shown geometrically to be incommensurable by the following diagram.

By the method of finding the greatest common unit of measure of two quantities, lay off the side $B\,C$ upon the diagonal $BA$. Erect a $\perp$ to $BA$ at $D$. By Exercise 80, $AD = DE = EC$. $AE$ is the diagonal of a square of which $AD$ is the side. Lay off $AD$ upon $BC$, the last divisor, or its equal $AC$. It is contained twice with a remainder, as $AO$. Erect a $\perp$ to $AC$ at $O$. $AO = OM = MD$. $AM$ is the diagonal of a square of which $AO$ is a side. $AO$ is contained twice in $AD$ with a remainder. By continuing the division in the same way, it can be seen that the last remainder is always contained twice in the last preceding divisor with a remainder, for each time the remainder is the difference between the side and the diagonal of a square. The conditions after each division are exactly the same as after the previous division; hence, like conditions must recur without limit; hence, the division can never be exact; hence, there is no common divisor between the side and diagonal of a square; hence, they are incommensurable.




(*b*) The diagonal and side of a square will later be proved to be in the ratio of $\sqrt{2}$ to 1, or $\frac{\sqrt{2}}{1}$, which equals $\sqrt{2}$. The $\sqrt{2}$ can be shown algebraically to be a surd; that is, incommensurable with the unit one.

**206.** The circumference of a circle and its diameter are incommensurable. The two lines which bear the relation of 1 and $\sqrt{3}$, or 1 and $\sqrt{5}$, are incommensurable. Many quantities in common experience are incommensurable, but as the remainder is small, it is ignored and the quantities are regarded as actually commensurable. In measuring distances the carpenter discards the remainder to the degree of accuracy that his work requires.

Show further illustrations of incommensurables in common experience.

207. **An incommensurable number** is a number which is incommensurable with its unit, as the $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{11}$, or 3.1416+ etc.

208. **An incommensurable ratio** is the ratio of any two incommensurable quantities or numbers. The side and diagonal of a square, § 205 (*a*), also the diameter and circumference of a circle, when compared, form an incommensurable ratio.

NOTE.—An incommensurable number must not be looked upon as an inexact number. If the side of a square is 1 foot, the diagonal is $\sqrt{2}$ feet, but this diagonal is a perfectly definite length, and can be constructed as exactly as any other line.

209. **A constant** is a quantity whose value is fixed.

210. **A variable** is a quantity which, under the conditions imposed upon it, may assume an indefinite number of values. For example, the distance from a railway station to a moving train is a quantity which has one value at one time, another value a minute later, and still different values later on.

211. When a variable so changes that it continually approaches a constant which it cannot reach, but from which it may be made to differ by an amount less than any assigned quantity, however small, the constant is the **limit of the variable**, and the variable approaches its limit.

NOTE.—By this definition some variables have limits and some do not. The variable in the illustration § 210 belongs to the latter class; the variables considered in geometry belong to the former.

(*a*) Tests for the limit of a variable:

1. Is it a constant?
2. Is it approached in value by a variable?
3. Can it be equaled by the variable?
4. Can the difference between the variable and its limit be made less than any assigned quantity?

(*b*) If a point is made to move along a given line, the distance from a fixed point to the moving point is a variable. This variable may or may not have a limit. If such conditions are imposed upon the moving point that, during a given period of time, it moves along half the length of the line, the next period along half the remaining distance, the next half the remaining distance, etc., the length of the given line is the limit of the variable.

(*c*) One $m$th of a given line is a variable which approaches zero as a limit if $m$ is made to increase indefinitely; $a$ minus one $m$th of $a$ is a variable which approaches $a$ as a limit if $m$ is made to increase indefinitely.

Illustrate freely the idea of variable in the article above; as, if 16 is divided by an increasing series of numbers, as 2, 4, 8, 16, 32, 64, 128, etc., the quotient is a variable approaching zero as its limit.

(*d*) The perimeter of a polygon consisting of equal chords of a circle is a variable, if the number of its sides is continually increased by drawing chords from the vertices of the polygon to the middle points of the arcs.

What is the limit of this perimeter?

Test your answer by the definition of limit.

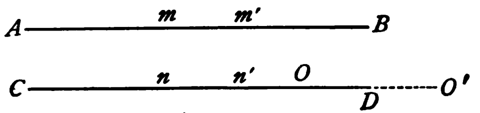
(*e*) The area of the polygon is also a variable under the above conditions. What is its limit?

(*f*) The pupil should remember that the limit and the variable are always the same kind of quantities; as in example (b), the limit and variable are both distances; in (d), are both lines, etc.

**212.** The difference between a variable and its limit is a variable whose limit is zero.

## PROPOSITION XVII.

**213. Theorem.** *If two variables are always equal as they approach their limits, their limits are equal.*




**Let A *m* and C *n* represent two equal variables, and A B and C D their respective limits.**

*To prove that A B equals C D.*

*A B* and *C D* are either equal or unequal. Suppose *A B* is smaller than *C D*. Let *A B*=*C O*. Lay off *A B* on *C D*; then *C O* becomes the limit of the variable *A m*.

By hypothesis *A m* = *C n*, *A m'* = *C n'*, etc. By definition of limit of a variable, *A m* approaches indefinitely near *C O* and *C n* approaches *C D*. Hence, as *O D* is an assigned quantity to denote the difference between *A B* and *C D*, the variable *A m* eventually becomes smaller than *C n* as they together approach their limits. This is contrary to the hypothesis, hence *A B* cannot be smaller than *C D*. In a similar manner prove *A B* cannot be larger than *C D*.

*Therefore*—

The absurdity of the assumption of the inequality of

$AB$ and $CD$ can be shown also in a violation of the definition of limit of a variable.

In the above demonstration, if the variables maintain their equality, one of them cannot approach within less than an assigned distance of its limit, or the other must become larger than its limit. If $Am$ remains equal to $Cn$, as $Cn$ approaches its limit $CD$, it must become larger than $CO$ its limit; which violates the definition of limit of a variable.

**214.** POSTULATES. (*a.*) *The variable which is obtained by multiplying a given variable by a constant has for its limit the product of the limit of the variable by the constant.*

If $V$ is a variable, $L$ its limit, and $A$ any constant, $A \times V$ approaches $A \times L$ as its limit.

The following may serve as a numerical illustration:

$$2 \times (.3, .33, .333 \ldots \tfrac{1}{3}) = .6, .66, .666 \ldots \tfrac{2}{3}.$$

(*b.*) *The variable which is obtained by dividing a given variable by a constant, has for its limit the quotient of the limit of the given variable by the constant.*

If $V$ is a variable, $L$ its limit, and $A$ any constant, $\frac{V}{A}$ approaches $\frac{L}{A}$ as its limit.

To illustrate: $(.6, .66, .666 \ldots \tfrac{2}{3}) \div 2 = .3, .33, .333 \ldots \tfrac{1}{3}$.

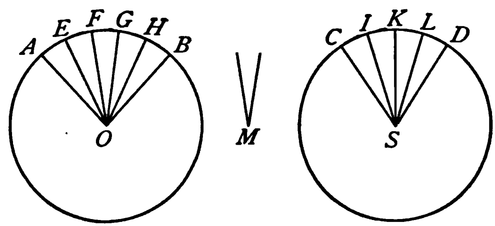
(*c.*) *If two variables, both increasing or both decreasing, are multiplied together term for term the product is a variable, the limit of which is the product of the limits of the variables.*

If $V$ is one variable, and $V'$ another, $L$ and $L'$ their respective limits, both variables increasing or both decreasing, $V \times V'$ is a variable, and $L \times L'$ is its limit.

To illustrate: Multiply the above numerical variables term for term.

## PROPOSITION XVIII.

**215. Theorem.** *In the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept at the circumference.*




**Let O and S represent angles at the center of two equal circles, and A B and C D the arcs which they intercept at the circumference.**

*To prove that* $\frac{\text{angle } O}{\text{angle } S}$ *is equal to* $\frac{\text{arc } AB}{\text{arc } CD}$.

There are two cases.

CASE I. *When angles O and S are commensurable.*

*Suggestion 1.* Let $\angle M$ represent a common unit of measure for the two angles.

*2.* If $\angle M$ is contained 5 times in $\angle O$ and 4 times in $\angle S$, what does the ratio $\frac{\angle O}{\angle S}$ equal? § 202.

Or, if $\angle M$ is contained *a* times in $\angle O$ and *b* times in $\angle S$, what does the ratio of $\frac{\angle O}{\angle S}$ equal?

*3.* Extend the lines of division of the $\angle$s to the arcs. How do arcs $AE$, $EF$, $CI$, $IK$, etc., compare? Why? Hence, one of them may be used as a unit of measure.

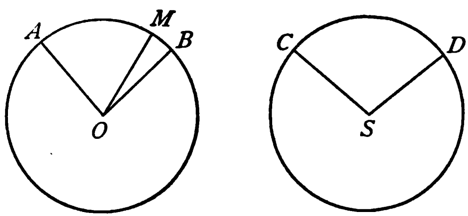
*4.* How many times is this unit arc contained in arc $A\,B$? In arc $C\,D$?

*5.* Then the ratio $\frac{arc\ A\,B}{arc\ C\,D}$ equals what? § 202.

*6.* Compare the ratio of the ∠s with the ratio of the arcs. Give auth.

*Therefore* —

CASE II. *When the angles O and S are incommensurable.*




*Suggestion 1.* Take any unit of measure of one of the angles, as of $\angle S$, and apply this unit to the $\angle O$. There must be a remainder. Why?

Let this remainder be represented by $\angle M\,O\,B$. How large is this remainder with respect to the unit of measure? Why?

*2.* The ∠s $A\,O\,M$ and $S$ are commensurable. Why?

*3.* Compare the ratio $\frac{\angle A\,O\,M}{\angle S}$ with the ratio $\frac{arc\ A\,M}{arc\ C\,D}$.
Case I.

*4.* By taking the unit of measure of $\angle S$ smaller and smaller continually, the remainder in $\angle A\,O\,B$, namely, the $\angle M\,O\,B$, may be made as small as we please, that is, it may be made to approach zero at its limit. (§ 212.) Can it be made to disappear entirely? Why? Sug. 1.

*5.* The $\angle A\,O\,M$ is a variable. Why?

*6.* Is $\angle S$ a variable or a constant?

*7.* Is the ratio $\frac{\angle AOM}{\angle S}$ a variable or constant? What

*ratio* is $\frac{\angle AOM}{\angle S}$ approaching? § 214 (*b*).

*9.* What kind of a quantity is arc $AM$? Is arc $CD$?

The ratio of $\frac{\text{arc } AM}{\text{arc } CD}$?

*10.* What is the limit of $\frac{\text{arc } AM}{\text{arc } CD}$? § 214 (*b*).

*11.* Compare the two variables $\frac{\angle AOM}{\angle S}$ and $\frac{\text{arc } AM}{\text{arc } CD}$ as they approach their limits. Sug. 3.

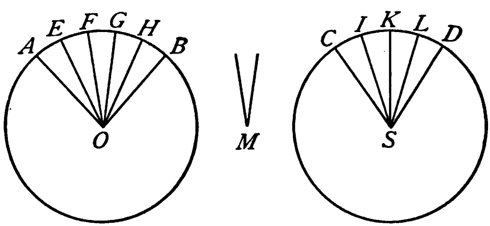
*12.* Compare their limits. § 213.

*Therefore —*

**Model.**

PROPOSITION XVIII.

**Theorem.** *In the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept at the circumference.*




**Let O and S represent angles at the centers of two equal circles, and A B and C D the arcs which they intercept at the circumference.**

*To prove that* $\frac{angle\ O}{angle\ S}$ *is equal to* $\frac{arc\ A\ B}{arc\ C\ D}$.

There are two cases.

CASE I. *When angles O and S are commensurable.*

Let $\angle M$ represent a common unit of measure of $\angle O$ and $\angle S$.

If $\angle M$ is contained 5 times in $\angle O$ and 4 times in $\angle S$, the ratio $\frac{\angle O}{\angle S} = \frac{5}{4}$. § 202.

Or, if $\angle M$ is contained $a$ times in $\angle O$ and $b$ times in $\angle S$, the ratio $\frac{\angle O}{\angle S} = \frac{a}{b}$. § 202.

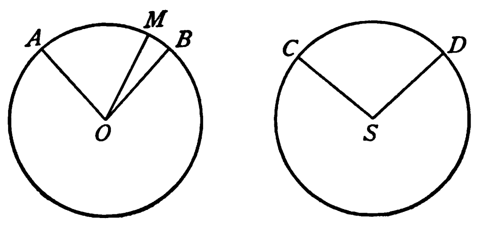
Extend the lines of division of the $\angle$s to the arcs; then arcs $A\ E$, $E\ F$, $C\ I$, $I\ K$, etc., are all equal, and hence the arcs $A\ B$ and $C\ D$ can be measured by using any one of them as a unit of measure. § 177, § 198

The unit arc is contained 5 times in arc $A\ B$ and 4 times in arc $C\ D$, or $a$ and $b$ times respectively, for each angle at the center intercepts an arc.

Hence, the ratio $\frac{arc\ A\ B}{arc\ C\ D} = \frac{5}{4}$ or $\frac{a}{b}$. § 202.

$\therefore \frac{\angle O}{\angle S} = \frac{arc\ A\ B}{arc\ C\ D}$. *Ax.* 1.

CASE II. *When angles O and S are incommensurable.*



Take a unit of measure of one of the $\angle$s, as of $\angle S$, and apply this unit to the $\angle O$. Since the angles are incommensurable, this unit will be contained in $\angle O$ a certain number of times with a remainder, as $\angle MOB$, which is less than the unit of measure. § 205.

Since the $\angle$s $AOM$ and $S$ have a common unit of measure, they are commensurable, and hence, by Case I,

$$\frac{\angle AOM}{\angle S} = \frac{\text{arc } AM}{\text{arc } CD}.$$

By taking the unit of measure of $\angle S$ smaller and smaller continually, the remainder in $\angle AOB$, viz., the $\angle MOB$, which is always less than the unit, may be made indefinitely small, but cannot be made entirely to disappear, for then the $\angle$s $O$ and $S$ would be commensurable. Hence, the remainder, $\angle MOB$, approaches zero as a limit, and, therefore, the $\angle AOM$ is a variable which approaches $\angle AOB$ as a limit. § 211.

But $\angle S$ is a constant, and hence, the ratio $\frac{\angle AOM}{\angle S}$ is a variable which approaches the ratio $\frac{\angle AOB}{\angle S}$ as a limit. § 214.

Arc $AM$, as $\angle AOM$ increases, is made to increase (§ 177), and hence is a variable, which approaches arc $AB$ as a limit (§ 211). Arc $CD$ is a constant, hence the ratio $\frac{\text{arc } AM}{\text{arc } CD}$ approaches the ratio $\frac{\text{arc } AB}{\text{arc } CD}$ as a limit. § 214.

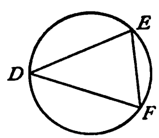
As $\frac{\angle AOM}{\angle S}$ and $\frac{\text{arc } AM}{\text{arc } CD}$ are two variables which are always equal as they approach their limits, their limits must be equal (§ 213). That is, $\frac{\angle AOB}{\angle CSD} = \frac{arc\ AB}{arc\ CD}$.

Therefore, in the same circle, or in equal circles, two angles at the center have the same ratio as the arcs which they intercept on the circumference.

**216.** Scholium I. A degree, which has already been defined as $\frac{1}{90}$ of a right angle, or $\frac{1}{360}$ of the whole angular magnitude about a point, is the standard unit angle; and the arc it intercepts on the circumference, also called a degree, or $\frac{1}{360}$ of the circumference, is the standard unit arc.

**217.** Scholium II. By the proposition it is proved that the number of times the unit angle is contained in the given angle at the center is the same as the number of times that the unit arc is contained in the arc intercepted by the given angle. Hence, we say briefly that **an angle at the center is measured by its intercepted arc,** meaning that there are as many degrees in the angle at the center as in the intercepted arc.

**218.** Scholium III. A right angle which contains 90 degrees (written 90°), is often used as a unit angle in determining the size of other angles.

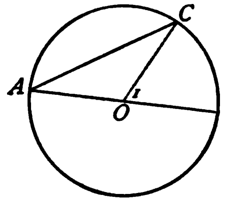



**219.** An angle **inscribed in a circle** is one whose vertex is in the circumference, and whose sides are chords of the circle. The angle $E$ is an inscribed angle.

**220.** An angle is **inscribed in a segment** of a circle when its vertex is in the arc of the segment and its sides meet the extremities of the chord of the arc. The angle $E$ is inscribed in the segment $DEF$.

## Proposition XIX.

221. **Theorem.** *An inscribed angle is measured by one-half its intercepted arc.*




**Let A be an inscribed angle, intercepting the arc B C.**

*To prove that the angle A is measured by one-half the arc B C.* § 217.

Case I. *When one side of the angle is a diameter.*

*Suggestion 1.* Connect $O$, the center of the circle, with $C$.

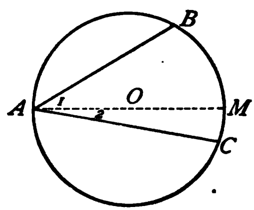
2. Compare $\angle A$ with $\angle C$. Give auth.

3. Compare $\angle A$ with $\angle 1$. Give auth.

4. By what arc is $\angle 1$ measured? Why? § 217.

5. Then, by what part of the arc $BC$ is the $\angle A$ measured?

Case II. *When the center lies between the sides of the angle.*




*Suggestion 1.* Through the vertex $A$ draw the diameter $AM$.

2. By what arc is $\angle 1$ measured? Why?

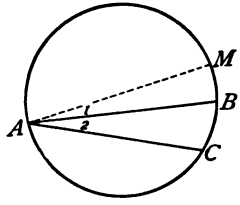
3. By what arc is $\angle 2$ measured? Why?

4. Then, by what arc is $\angle A$ measured?

NOTE.—Suggestion 4 may be more easily comprehended if the definition of "measured" (§ 217) be substituted; as, the number of angle degrees in $\angle 1$ plus the number of angle degrees in $\angle 2 = \frac{1}{2}$ the number of arc degrees in $B M$ plus $\frac{1}{2}$ the number of arc degrees in $M C$.

Express algebraically and reduce by factoring.

CASE III. *When both sides of the angle are on the same side of the center.*




*Suggestion 1.* Through the vertex $A$ draw the diameter $A M$.

2. By what arc is $\angle A$ measured?

3. By what arc is $\angle 1$ measured?

4. Then, by what arc is $\angle 2$ measured? (See Note, Case II.)

*Therefore—*

**222.** COROLLARY I. *An inscribed right angle is measured by one-half a semi-circumference.*

**223.** COROLLARY II. *An angle inscribed in a semi-circle is a right angle.*

QUERY. How could you use Corollary II to construct a right angle?

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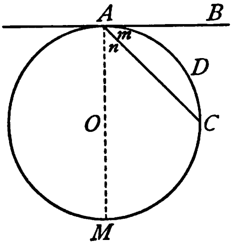
Ex. 127. If angle $A$, § 221, Case I $= 63\frac{1}{2}°$, how many degrees in angle 1? In angle $C$? In arc $A C$?

## PROPOSITION XX.

**224. Theorem.** *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*




**Let $m$ represent an angle formed by A B a tangent, A C a chord, and A D C the intercepted arc.**

*To prove that the angle $m$ is measured by one-half the arc $A\ D\ C$.*

*Suggestion 1.* Draw through $A$ the diameter $A\ M$. What kind of an ∠ is $B\ A\ M$? Why? By what arc is it measured? § 95.

*2.* By what arc is ∠ $n$ measured? Why?

*3.* By what arc is ∠ $m$ measured? Why? (See Note § 221.)

*Therefore—*

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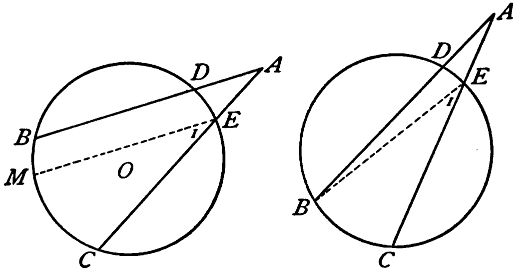
Ex. 128. A chord is met at its extremity by a tangent forming an angle of 75°. How many degrees in the arc that is subtended by the chord?

Ex. 129. $a$, $b$ and $c$ are the angles of an inscribed triangle. Angle $a$ is four times $b$, and $b$ is one-seventh of $c$. How many degrees in the arcs of the circle subtended by the sides of the angles, respectively?

## Proposition XXI.

**225. Theorem.** *The angle formed by two secants, meeting without the circle, is measured by one-half the difference of the intercepted arcs.*




**Let A B and A C represent two secants, meeting at A, without the circle, forming the angle B A C and intercepting the arcs D E and B C.**

*To prove that the angle B A C is measured by one-half the difference of the arcs B C* and *D E.*

*Suggestion 1.* Through *E*, draw a chord *E M* ‖ to *A B*.

*2.* By what arc is ∠ *1* measured? Why?

*3.* Compare the ∠ *A* with ∠ *1*.

*4.* Then, by what arc is ∠ *A* measured?

*5.* Express the arc *M C* in terms of *B C* and *B M*, then in terms of *B C* and *D E*.

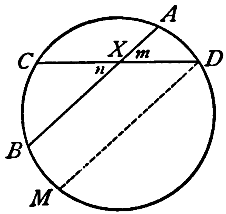
*6.* Then, by what arc is ∠ *A* measured?

*Therefore—*

Another Method. Connect *B* and *E*. Compare ∠ *A* with ∠ *B* and ∠ *1*. Complete the demonstration.

## Proposition XXII.

**226. Theorem.** *An angle formed by two intersecting chords is measured by one-half the sum of the intercepted arcs.*




**Let A B and C D represent two chords intersecting at X.**

*To prove that angle n is measured by one-half the sum of arcs C B and A D.*

*Suggestion 1.* Through *D*, one extremity of the chord *C D*, draw *D M* ∥ to *A B*.

*2.* Compare ∠ *n* with ∠ *D*. Give auth.

*3.* By what arc is ∠ *D* measured? Why?

*4.* By what arc is ∠ *n* measured?

*5.* Express the arc *CM* in terms of *BC* and *A D*. § 193.

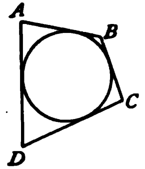
*6.* Then, by what arc is ∠ *n* measured in terms of *BC* and *A D*?

*Therefore*—

Another Method. Connect *B* and *D*. Compare ∠ *n* with ∠ *B* and ∠ *D*. Complete the demonstration.

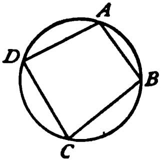
Query. What angles have we learned to measure? What is the measure of each?

**227.** A polygon is **circumscribed about a circle,** when each of its sides is tangent to the circle; as polygon $A\,B\,C\,D$.



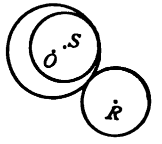

(*a*) When a polygon is circumscribed about a circle, the circle is **inscribed in the polygon.**

**228.** A polygon is **inscribed in a circle** when each of its sides is a chord of the circle; as polygon $A\ C$.




(*a*) When a polygon is inscribed in a circle, the circle is **circumscribed about the polygon.**

**229.** Two **circles are tangent** to each other when they have one point of contact, and only one, as circles $O$ and $R$, or $O$ and $S$.




(*a*) This point of contact is called the **point of tangency.**

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Ex. 130. An angle formed by two tangents is measured by one-half of the difference between the intercepted arcs.

Ex. 131. An angle formed by a tangent and a secant is measured by one-half of the difference between the intercepted arcs.

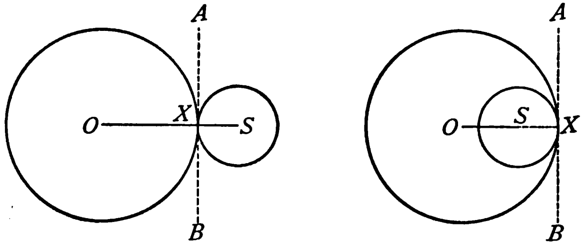
Ex. 132. If the middle points of the sides of a quadrilateral be joined in order, the figure formed is a parallelogram. *Sug.* Draw the diagonals of the quadrilateral.

Ex. 133. The diameters of any quadrilateral bisect each other.

Ex. 134. If $A$ is the vertex and $B\,C$ the base of an isosceles triangle $A\,B\,C$, and if from any point $D$ in the side $A\,B$ a line is drawn perpendicular to the base and meeting $C\,A$ extended at $E$, prove that the triangle $A\,D\,E$ is isosceles.

## PROPOSITION XXIII.

230. **Theorem.** *If two circles are tangent to each other, the line joining their centers passes through the point of tangency.*




**Let circles O and S be tangent to each other, X the point of tangency and O S the line joining the centers.**

*To prove that O S, extended if necessary, passes through the point of tangency X.*

*Suggestion 1.* Represent a straight line *A B* that is a tangent common to the two circles.

*2.* At the point of tangency *X*, erect a ⊥ to *A B*, and extend it through both circles.

*3.* The ⊥ passes through the centers. Why?

*4.* What relation does this ⊥ sustain to *O S*? Give auth. *Ax. 12.*

*Therefore—*

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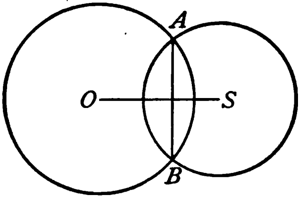
Ex. 135. *A B* is the hypotenuse of a right triangle *A B C*, *B D* is drawn bisecting the angle *B*, meeting *A C* at *D*, and *D E* is drawn perpendicular to *A C*, meeting *A B* at *E*. Prove that *E D B* is an isosceles triangle.

## PROPOSITION XXIV.

**231. Theorem.** *If two circles intersect, the line joining their centers is perpendicular to their common chord at its middle point.*




**Let circles O and S intersect. Let O S be the line joining their centers and A B their common chord.**

*To prove that O S is perpendicular to A B at the middle point of A B.*

*Suggestion.* At the middle point of *A B* erect a ⊥ and extend both ways. (§ 186.) Complete the demonstration.

*Therefore—*

ANOTHER METHOD:

*Suggestion 1.* What is the locus of points equally distant from *A* and *B*?

*2.* What relation do the centers bear to *A* and *B*?

*3.* What relation must they bear to the locus?

*Therefore—*

**232.** COROLLARY I. *If two intersecting circles are equal, the line of centers is bisected by the common chord.*

*To prove O S is bisected.*

QUERY. How could § 232 be used to bisect a straight line?

### 233. Review.

I. In the same circle or in equal circles:

(*a*) If two angles at the center are equal, compare the subtending arcs; the subtending chords.

(*b*) If two angles at the center are unequal, compare the subtending arcs; the subtending chords.

(*c*) If two arcs are equal, compare the subtended angles at the center; the subtending chords.

(*d*) If two arcs are unequal, compare the subtended angles at the center; the subtending chords.

(*e*) If two chords are equal, compare the arcs they subtend; the angles at the center they subtend.

(*f*) If two chords are unequal, compare the arcs they subtend; the angles at the center they subtend.

II. What lines cut the centers of circles?

III. What angles have been measured? What arcs are the respective measures of the angles?

### Problems of Construction.

**234.** In the previous work the constructions have been represented, instead of being actually performed. If it has been established that a certain relation of points, lines or surfaces is possible, the rigor of the demonstration is not impaired by using a representation of that relation without actually constructing it.

**235.** Since plane geometry deals only with figures which can be made from straight lines and circumferences of circles, the constructions of plane geometry are those which it is possible to effect by means of a straight edge and dividers, or compasses, which are necessitated by the postulates named below.

Problems of construction belong in no sense to pure geometry, but are applications of the principles demonstrated in pure geometry.

**236.** Since constructions are based upon theorems previously demonstrated, if several theorems bearing upon the same point have been established, more than one method of solution of the problem involving that point can usually be effected; as, if the problem be given to draw through a given point a line parallel to a given line, several methods for the solution of the problem can be evolved by means of several of the propositions concerning parallel lines, as §§ 114–116.

### Postulates of Construction.

**237.** Let it be granted:

(*a*) That a straight line can be drawn between any two points and can be extended to any length through either extremity.

(*b*) That a circle can be drawn with any point as a center, and with any straight line as a radius.

NOTE.—It will be observed that the first postulate necessitates the straight edge, and the second the dividers.

---

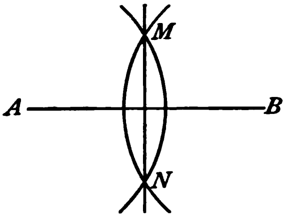
Ex. 136. If the middle points of the three sides of a triangle be joined by straight lines, the triangle is divided into four triangles, which are equal in all respects.

Ex. 137. Two circles are tangent internally. Two lines are drawn from the point of tangency through the extremities of the diameter of one of the circles. Prove that they intersect the other circle in the extremities of a diameter.

Ex. 138. Prove Ex. 137 is true if the circles are tangent externally.

## PROPOSITION XXV.

**238. Problem.** *To bisect a given straight line.*




**Let A B be given a straight line.**

*To bisect A B.*

*Suggestion 1.* Use § 232.

*2.* To use this truth, *A* and *B* must be the centers of two equal intersecting circles. Hence, with these points as centers, construct two equal intersecting circles (§ 237 (*b*)), and draw their common chord *M N* (§ 237 (*a*) ).

*Therefore—*

What postulates are used?

Try § 231 for another solution.

QUERY. At what angle does *M N* bisect *A B*?

NOTE.—A little experience will suggest how to omit the unessential parts of lines. For instance, in the above construction all the circumferences can be omitted, except short arcs near the points of intersection, *M* and *N*.

**239.** The **median** of a trapezoid is the straight line joining the middle points of the legs.

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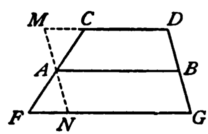
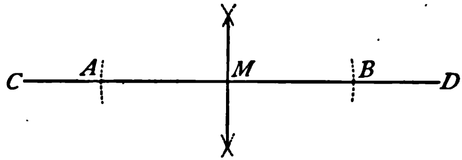



Ex. 139. The sum of the bases of a trapezoid is equal to twice the median of the trapezoid.

Ex. 140. The median of a trapezoid is parallel to the bases.

## PROPOSITION XXVI.

**240. Problem.** *To erect a perpendicular to a given line, at a given point in that line.*




**Let C D be the given line and M the given point.**

*To erect a perpendicular to C D at M.*

*Suggestion 1.* Use § 232.

*2.* In order to use this principle, *M* must be the middle point of the line joining the centers of the ⊙s. Hence, find points on the line *C D*, as *A* and *B*, equally distant from *M* (§ 237 (*b*) ), and proceed as in § 238.

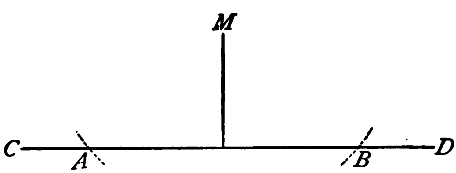
*3.* A perpendicular is erected to *C D* at *M*.

*Therefore—*

For other methods, see §§ 185, 231, Ex. 29.

## PROPOSITION XXVII.

**241. Problem.** *From a given point, to drop a perpendicular to a given line.*




**Let C D represent the given line, and M the given point.**

*To drop a perpendicular from M to the line C D.*

*Suggestion 1.* As $M$ is one point in the required $\perp$, but one other point has to be determined. *Ax.* 12, Cor. 2.

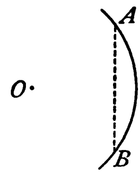
2. Effect a construction so that you can use the truth in § 185.

*Therefore—*

QUERY. What postulates and problems are used in this problem? §§ 237 (*b*), 238 and 237 (*a*).

PROPOSITION XXVIII.

**242. Problem.** *To bisect a given arc.*




**Let A B represent the given arc.**

*To bisect the arc A B.*

*Suggestion 1.* What propositions have been demonstrated which involve the bisection of an arc? §§ 184, 185, 187.

2. If the center, as $O$, is given, apply § 184 or § 185 and complete the solution.

3. If the center is not given, apply § 187.

*Therefore—*

---

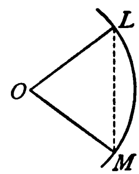
Ex. 141. Let $a$ be a given straight line, and $b$ an indefinite straight line intersecting it. Construct a right triangle having $a$ for its hypotenuse and having its vertex in $b$. § 223.

Ex. 142. $A\,B$ and $C\,D$ are two chords of a circle intersecting at $O$. Prove triangles $A\,O\,D$ and $C\,O\,B$ are mutually equiangular. Prove triangles $A\,O\,C$ and $B\,O\,D$ mutually equiangular.

### PROPOSITION XXIX.

**243. Problem.** *To bisect a given angle.*

**Let O represent a given angle.**




*To bisect the angle O.*

*Suggestion 1.* What propositions involve the bisection of an ∠? Make the construction by one or more methods.

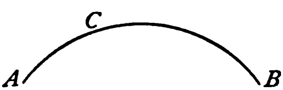
§§ 242, 178, Ex. 54, etc.

*Therefore—*

### PROPOSITION XXX.

**244. Problem.** *To find the center of a circle, if any arc of the circumference is given.*

**Let A B represent an arc.**




*To find the center of a circle of which A B is an arc.*

*Suggestion 1.* How many lines must pass through a given point to determine it?

*2.* What principles have been demonstrated which determine lines that pass through the center of a ⊙?

§§ 186, 192, Ex. 114, etc.

*3.* Use § 186 to determine two loci of the center.

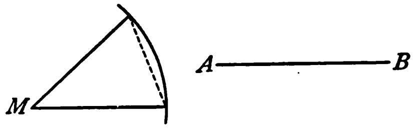
*4.* Why is their point of intersection the required center? § 195.

*Therefore—*

NOTE.—Only those truths previously proved that involve postulates, or problems that have been solved, can be used in the solution of a problem. For example, § 186 leads to a possible solution of the problem, § 244, because the problem to erect a perpendicular has already been solved. Can § 192 be used? Why? What is the objection to the use of Ex. 114?

### PROPOSITION XXXI.

**245. Problem.** *At a given point in a given line, to construct an angle equal to a given angle, with the given line as one side.*




**Let M represent the given angle, A B the given line, and A the given point.**

*To construct at A an angle equal to the angle M, having A B for one side.*

*Suggestion 1.* Use §§ 173, 181, and 178.

*2.* To apply these truths, *M* and *A* must be made centers of = ⊙s. How can this be done?

*3.* Equal arcs must be made to subtend ∠ *M* and the ∠ to be constructed at *A*. How can this be done? Complete the construction.

*Therefore*—

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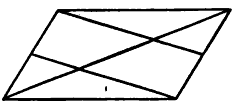
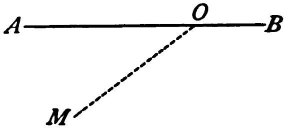
Ex. 143. If from two opposite vertices of a parallelogram two lines be drawn to the middle points of two opposite sides, the lines will trisect the diagonal joining the other vertices. (See Ex. 101.)



Ex. 144. *A B C* is a triangle inscribed in a circle whose center is *O*. *O D* is perpendicular to *B C*. Prove angle *D O C*, or its supplement, is equal to angle *A*. *Sug.*: *O* may be within or without the triangle.

## Proposition XXXII.

**246. Problem.** *Through a given point, to construct a straight line parallel to a given straight line.*




**Let A B be a given straight line, and M a given point.**

*To construct a straight line through M, parallel to A B.*

*Suggestion 1.* What propositions determine that lines are ∥ ? §§ 102, 114, 115, etc.

*2.* These truths require a transversal through *M*, cutting *A B*. Use § 114. Use § 102. Complete the constructions.

Other Methods. If time permits, try to use other truths by which a line ∥ to another line is determined; as, for instance, §§ 152 and 140 (*c*).

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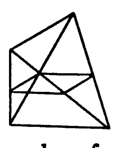
Ex. 145. If the middle points of two opposite sides of a quadrilateral be joined to the middle points of the diagonals, the joining lines form a parallelogram.



Ex. 146. Prove that the three bisectors of the angles of a triangle meet in the same point.

*Suggestion 1.* In triangle *A B C* the bisector of ∠ *A* is the locus of a point equally distant from the sides *A B* and *A C*. Ex. 71.

*2.* The bisector of ∠ *B* is the locus of what?

*3.* Prove, now, that the point of intersection of these two bisectors is on the bisector of the ∠ *C*.

## PROPOSITION XXXIII.

**247. Problem.** *To construct* (1) *the complement of a given angle*; (2) *the supplement of a given angle;* (3) *the third angle of a triangle, having given the other two.*

*Suggestion 1.* See definitions of complement and supplement of an angle.

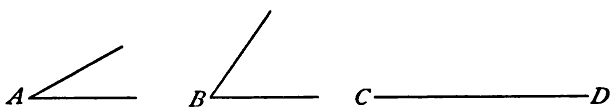
*2.* In solving (1), § 240.

*3.* In solving (3), § 51 or § 120.

*Therefore—*

## PROPOSITION XXXIV.

**248. Problem.** *Given two angles and the included side of a triangle, to construct the triangle.*




**Let A and B represent the two given angles, and C D the included side.**

*To construct a triangle having A and B for two of its angles, and C D the side included between these angles.*

*Suggestion.* Represent the △ as if already constructed. Study it to see what previous problems need to be employed in the construction.

QUERY. Could the line and angles be of such magnitudes that the construction of a triangle is impossible?

## Proposition XXXV.

**249. Problem.** *Given two sides and the included angle of a triangle, to construct the triangle.*

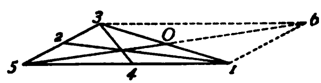

**Let M and N represent two sides and O the included angle of a triangle.**

*To construct a triangle having M and N for two of its sides, and O the angle formed by those two sides.*

*Suggestion.* See Sug. § 248.

---

Ex. 147. The three medians of a triangle meet at a point. (See Ex. 143.) *1 2* cuts off one-third of diagonal *5 6*. Find relation of *3 4* to diagonal. Show that median *5 O* lies in diagonal *5 6*.




Ex. 148. Prove that the line which bisects an arc and is perpendicular to its subtending chord passes, if extended, through the center of the circle.

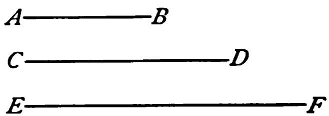
Try this example in the construction of problem, § 244.

Ex. 149. Parallel straight lines, included between parallel straight lines, are equal.

Ex. 150. If a line be drawn in a trapezoid, bisecting one of the legs and parallel to the bases, prove that it bisects the other leg also. *Sug.:* Draw a diagonal of the trapezoid.

## Proposition XXXVI.

**250. Problem.** *To construct a triangle whose sides are three given lines.*




**Let A B, C D, and E F represent three given lines.**

*To construct a triangle whose sides are A B, C D, and E F, respectively.*

*Suggestion 1.* Draw one line as *A B*. Two vertices of the △ are determined. The problem now is to determine the third vertex, *X*.

*2.* Having constructed line *A B*, what is one locus of the vertex? § 158 or Ex. 100.

*3.* What is another locus of the vertex?

*4.* Do these loci intersect?

*5.* Having located point *X*, construct the △.

Query. Could *C D* and *E F* be of such length with respect to *A B* that the loci could not intersect? Discuss the possibility of the solution as dependent upon the relative lengths of the three given lines. § 87.

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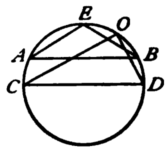
Ex. 151. Upon a given base, construct an isosceles triangle, in which the sum of the two equal sides equals a given line.

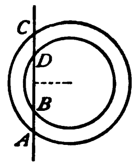
Ex. 152. All angles inscribed in the same segment are equal.

Ex. 153. In the same circle or in equal circles, an angle inscribed in the smaller of two segments is larger than an angle inscribed in the larger segment.




*Suggestion.* Prove that the angle *A E B* is larger than the angle *C O D*.

Ex. 154. The segments of a straight line intercepted by concentric circles are equal.

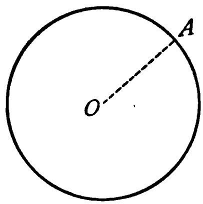



Prove *A B* equals *C D*.

## PROPOSITION XXXVII.

**251. Problem.** *Through a given point, to draw a tangent to a given circle.*

CASE I. *When the point is on the circumference.*



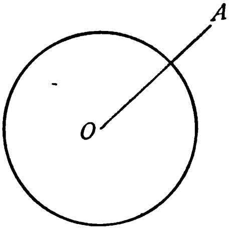

**Let A represent the given point in the circumference of the given circle, O.**

*To draw a tangent to the circle O, through the point A.*

*Suggestion 1.* What relation does a tangent bear to the radius drawn to the point of contact? Give auth.

What problems and postulates are involved in making application of the authority you use?

CASE II. *When the given point is without the circle.*




**Let A represent the given point without the circle O.**

*To draw a tangent to the circle O, through the point A.*

*Suggestion 1.* The problem is to determine the point of tangency. Draw line $OA$.

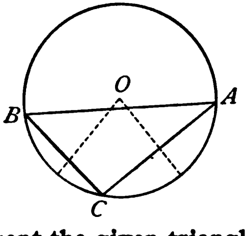
*2.* If the required point were connected with both $O$ and $A$, what kind of an ∠ would be formed at that point? Give auth.

*3.* § 189.

QUERY. How many tangents can be constructed from a point to a circle?

## PROPOSITION XXXVIII.

**252. Problem.** *To circumscribe a circle about a given triangle.*




**Let A B C represent the given triangle.**

*To circumscribe a circle about the triangle A B C.*

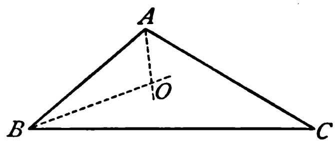
*Suggestion.* The problem is to find the center of a ⊙ whose circumference passes through *A*, *B*, and *C*; *i. e.*, to find a point equally distant from *A*, *B*, and *C*.

§ 137 or 195.

Query. How many circles can be circumscribed about a triangle? Why?

## Proposition XXXIX.

**253. Problem.** *To inscribe a circle in a given triangle.*




**Let A B C represent the triangle.**

*To inscribe a circle in the triangle A B C.*

*Suggestion 1.* Represent a ⊙ inscribed in a △. Study it.

§ 227.

*2.* What is the locus of a point equally distant from *A B* and *A C*? From *A B* and *B C*? §§ 131, 190, and Ex. 71.

*3.* Give complete directions for inscribing a ⊙ in a △.

Query. How many circles can be inscribed in a given △? Why?

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Ex. 155. If tangents to a circle be drawn at the extremities of any chord, these tangents make, with each other, an angle which is twice the angle between the chord and the diameter of the circle drawn through the extremity of the chord.

# CHAPTER III.

## PROPORTIONAL LINES, AND SIMILAR POLYGONS.

### The Theory of Proportion.

**254. A proportion** is an equality between ratios.

(*a*) Let $\frac{A}{B}$ represent a ratio and $\frac{C}{D}$ an equal ratio, then $\frac{A}{B}=\frac{C}{D}$ is a proportion.

(*b*) It may be written in the above form; as $A : B :: C : D$, or, as $A : B = C : D$, and is read: the ratio of $A$ to $B$ equals the ratio of $C$ to $D$; or, $A$ is to $B$ as $C$ is to $D$.

The reading, $A$ is to $B$ as $C$ is to $D$, belongs especially to the form, $A : B :: C : D$.

**255.** The **terms** of a proportion are the four numbers, or quantities, compared.

**256.** In any ratio the first term is the **antecedent,** and the second term the **consequent.** Hence, in any proportion the first and third terms are antecedents and the second and fourth terms are consequents.

**257.** In any proportion, the first and fourth terms are the **extremes,** and the second and third terms are the **means.**

In the proportion $\frac{A}{B}=\frac{C}{D}$, the first term is $A$, the second $B$, the third $C$, and the fourth $D$. $A$ and $C$ are antecedents, $B$ and $D$ are consequents, $A$ and $D$ are the extremes, and $B$ and $C$ are the means.

**258.** The terms of a ratio must be numbers, or quantities of the same kind. §§ 198 and 202.

**259.** A **fourth proportional** to three numbers or quantities is the fourth term of a proportion in which the three terms are the three quantities taken in order. $X$ is a fourth proportional to $A$, $B$, and $C$, if $\frac{A}{B} = \frac{C}{X}$. 12 is a fourth proportional to 6, 9, and 8 if $\frac{6}{9} = \frac{8}{X}$. Verify the last statement.

## Proposition I.

**260. Theorem.** *In a proportion, the product of the means equals the product of the extremes.*

Case I. *If all the terms are numbers.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a proportion in which A, B, C, and D are numbers.**

*To prove* $A D = B C$.

*Suggestion 1.* What process must be performed upon the ratio $\frac{A}{B}$ to produce the product $A D$?

*2.* What, then, must be done to the ratio $\frac{C}{D}$? Why?

*3.* What is the effect?

*Therefore* —

**261.** Case II. *If the terms are geometric magnitudes.*

As stated in this case, the theorem is impossible. However, the statement may be allowed to stand with a proper interpretation, viz., **If the terms of a proportion are geo-**

**metric magnitudes, the product of the measures of the means equals the product of the measures of the extremes.**

In the proportion $\frac{A}{B} = \frac{C}{D}$, let $A$, $B$, $C$, and $D$ represent geometric magnitudes. Let $M$ be a unit of measure of $A$ and $B$, and $N$ a unit of measure of $C$ and $D$. § 200.

*Suggestion 1.* If $M$ is contained in $A$ $s$ times and in $B$ $o$ times, what does the ratio of $A$ to $B$ equal? Why? § 202.

*2.* In the same way find the measures of $C$ and $D$. Find the ratio of $C$ to $D$ in terms of their measures.

*3.* Compare the ratios of the measures, and apply Case I.

*Therefore —*

**Model.**

*In a proportion, the product of the means equals the product of the extremes.*

CASE I. *If all the terms are numbers.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a proportion in which A, B, C, and D are numbers.**

*To prove* $A D = B C$.

$\frac{A}{B} \times B D = A D$, and $\frac{C}{D} \times B D = B C$.

$\therefore A D = B C$. *Ax.* 4.

CASE II. *If the terms are geometric magnitudes.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a proportion in which A, B, C, and D are geometric magnitudes. Let M be a unit of measure for A and B, and N a unit of measure for C and D.**

*To prove $A D = B C$ with the above interpretation.*

If $A$ contains $M$ $s$ times, and $B$ contains it $o$ times, then

$\frac{A}{B} = \frac{s}{o}$. § 202.

If $C$ contains $N$ $u$ times, and $D$ contains it $v$ times, then

$\frac{C}{D} = \frac{u}{v}$. $\therefore \frac{s}{o} = \frac{u}{v}$. *Ax.* 1.

$\therefore s v = o u$. Case I.

$s$ and $v$ are the measures of the extremes, and $o$ and $u$ are the measures of the means. § 200.

*Therefore —*

*In a proportion the product of the means equals the product of the extremes.*

**262.** NOTE.—It frequently occurs that mathematical expressions are used which are intelligible under some interpretations and have no meaning under others. For example, the expression $A B$ has a definite meaning, if $A$ and $B$ are both numbers; it also has a definite meaning if $A$ is a pure number and $B$ is a quantity, provided $A$ is considered as the multiplier; but if $A$ and $B$ are both quantities, $A B$ has no meaning. Likewise, the expression $\frac{A}{B}$ has a definite meaning if $A$ and $B$ are both numbers; it also has a definite meaning if $A$ is a quantity and $B$ a number; also, if $A$ and $B$ are both quantities of the same kind; but has no meaning if $A$ is a number and $B$ is a quantity, or if $A$ and $B$ are quantities unlike in kind. § 202 (1).

In all the following operations upon ratio, and their applications, care should be taken to interpret the symbols of number or quantity, to see whether the principles underlying the fundamental rules of arithmetic and algebra can be applied. The scholia following the theorems in the theory of proportion suggest the limitations that the principles underlying the operations impose upon the propositions.

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Ex. 156. Any straight line drawn through the point of intersection of the diagonals of a parallelogram divides the parallelogram into two parts which are equal in all respects.

## Proposition II.

**263. Theorem.** *If the product of two numbers is equal to the product of two other numbers, the factors of one product may be made the means, and the factors of the other product the extremes of a proportion.*

**Let A B = C D be an equation in which A, B, C, and D are numbers.**

*To prove* $\frac{A}{D} = \frac{C}{B}$, *or* $\frac{C}{A} = \frac{B}{D}$.

*Suggestion 1.* What process must be performed upon $A\,B$ to produce the fraction $\frac{A}{D}$?

2. See method in § 260.

*Therefore* —

**264.** Corollary I. *If a, b, c, and d in the equation* $a\,b = c\,d$ *be measures respectively of the quantities M, N, O, and P,*

*To prove that* $\frac{M}{O} = \frac{P}{N}$.

*Suggestion 1.* $\frac{a}{c} = \frac{d}{b}$. Why?

2. $\frac{a}{c} = \frac{M}{O}$. Why? $\frac{d}{b} = \frac{P}{N}$. Why? § 202.

3. $\frac{M}{O} = \frac{P}{N}$. Why?

*Therefore* —

**265.** A **mean proportion** is a proportion in which the means are the same or identical numbers or quantities.

**266.** A **mean proportional** between two numbers or quantities is the second or third term of a mean proportion.

$\frac{M}{N}=\frac{N}{O}$ is a mean proportion and $N$ is a mean proportional.

**267.** A **third proportional** is the fourth term of a mean proportion. $O$ is the third proportional, § 266.

## Proposition III.

**268. Theorem.** *A mean proportional equals the square root of the product of the extremes of the proportion.*

**Let** $\frac{\mathbf{A}}{\mathbf{B}}=\frac{\mathbf{B}}{\mathbf{C}}$.

*To prove that* $B=\sqrt{A\,C}$, in which $A\,C$ means $A\times C$.

Case I. *If $A$, $B$, and $C$ are numbers.*

*Suggestion*: In the equation $\frac{A}{B}=\frac{B}{C}$, solve algebraically to find the value of $B$.

*Therefore* —

Case II. *If $A$, $B$, and $C$ are geometric quantities.*

The theorem is not strictly true as stated, but needs interpretation; viz., the measure of the mean proportional equals the square root of the product of the measures of the extremes of the proportion.

*Suggestion*: Demonstrate by using method in § 261.

Note.—In dealing with proportions, the student should determine whether the terms are numbers or quantities, and observe the laws that govern them.

The expression, "the product of A and B," will be used indiscriminately, whether A and B are numbers or quantities. because of its brevity. The student must determine from the context whether it is a literal statement of fact, or is used in place of the expression, "the product of the measures of A and B." See § 262.

## PROPOSITION IV.

**269. Theorem.** *If in several successive ratios the consequent of the first is equal to the antecedent of the second, the consequent of the second is equal to the antecedent of the third, etc., then the ratio of the antecedent of the first ratio to the consequent of the last is equal to the product of the ratios.*

**Let $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, $\frac{D}{E}$, etc., represent the ratios.**

*To prove that* $\frac{A}{E}$ *is equal to the product of the ratios.*

CASE I. *If the terms of the ratios are numbers.*

*Suggestion 1.* If $A, B, C, D, E$, etc., are numbers, apply the algebraic rule for multiplying fractions.

*Therefore —*

CASE II. *If the terms of the ratios are geometric magnitudes.*

*Suggestion 1.* Why must all of the magnitudes be of the same kind? § 202.

Let $\frac{A}{B} = m$, $\frac{B}{C} = n$, $\frac{C}{D} = r$, $\frac{D}{E} = s$; $m, n, r$, and $s$ being numbers. § 200.

2. Then, $A = m\,B$, $B = n\,C$, $C = r\,D$, $D = s\,E$. Why? § 270.

3. Since $A = m\,B$ and $B = n\,C$, $A = m\,n\,C$. Why?
Since $A = m\,n\,C$ and $C = r\,D$, $A = m\,n\,r\,D$.

In the same way $A = m\,n\,r\,s\,E$.

4. Since $A = m\,n\,r\,s\,E$, $\frac{A}{E} = m\,n\,r\,s$. Why?

5. But $m\,n\,r\,s$ is the product of the given ratios.

*Therefore* —

**270.** $\frac{A}{B} = m$ means that $A$ contains the unit $B$, $m$ times.

This is only another form of expression for the statement that $A$ is equal to $m$ times the unit $B$, which in algebraic language is $A = m\,B$.

To illustrate: $\frac{1 \text{ bushel}}{1 \text{ peck}} = 4$, is another way of saying 1 bushel = 4 pecks. Page VIII.

NOTE.—A careful interpretation of the symbols should be made at each step in the preceding proposition. The proposition may be illustrated by any problem in reduction descending in denominate numbers; as, to reduce 1 bushel to pints. The table used, expressed in ratio form, is: $\frac{1 \text{ bushel}}{1 \text{ peck}} = 4$, $\frac{1 \text{ peck}}{1 \text{ quart}} = 8$, and $\frac{1 \text{ quart}}{1 \text{ pint}} = 2$. Performing the reduction, 1 bushel $= 4 \times 8 \times 2$ pints $= 64$ pints, or $\frac{1 \text{ bushel}}{1 \text{ pint}} = 64$.

QUERY: In the equation $A = m\,n\,r\,D$, which symbols represent numbers and which represent quantities?

---

Ex. 157. Determine a point at a given distance from a given point, and equally distant from two parallel lines.

How many points answer the conditions of the problem?

Is this problem always possible?

Ex. 158. Determine a point at a given distance from a given point, and equally distant from two given intersecting lines.

Show when there are four points, when three, when two, when one, and when not any.

## PROPOSITION V.

**271. Theorem.** *Both terms of a ratio can be multiplied by any number without changing the value of the ratio.*

**Let $\frac{A}{B}$ represent a ratio.**

*To prove* $\frac{A}{B} = \frac{mA}{mB}$, *m being a number.*

*Suggestion 1.* Let the ratio $\frac{A}{B} = r$. Then $A =$ what? § 270.

*2.* Find value of $mA$. Of $\frac{mA}{mB}$. Give auth.

*3.* Compare $\frac{mA}{mB}$ and $\frac{A}{B}$.

*Therefore —*

**272.** COROLLARY.—*Both terms of a ratio can be divided by the same number without changing the value of the ratio; also, both terms can be divided by the same quantity, provided it is a quantity of the same kind as the terms of the given ratio.*

**273.** SCHOLIUM.—In the foregoing proposition $A$ and $B$ may both be numbers, or may be like quantities.

---

Ex. 159. If one circle is inscribed in a right triangle, and another circle circumscribed about the same right triangle, the sum of the diameters of the circles is equal to the sum of the two legs of the right triangle.

Ex. 160. What is the locus of the middle points of all the chords of a circle which are parallel to a given line? Ex. 116.

## PROPOSITION VI.

**274. Theorem.** *If four numbers, or like quantities, are in proportion, the ratio of the first to the third equals the ratio of the second to the fourth.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.**

*To prove* $\frac{A}{C} = \frac{B}{D}$.

*Suggestion 1.* Let $\frac{A}{B} = m$. What does $\frac{C}{D} = ?$

*2.* What is the value of $A$? Of $C$?

*3.* Find the value of $\frac{A}{C}$ and reduce to simplest form.

*Therefore* —

**275.** SCHOLIUM.—Proposition VI can be applied only when the four terms are numbers or like quantities. For instance, $\frac{5 \text{ rods}}{10 \text{ rods}}$ may equal $\frac{8 \text{ lbs.}}{16 \text{ lbs.}}$ but to apply Proposition VI would necessitate the unthinkable expression $\frac{5 \text{ rods}}{8 \text{ lbs.}} = \frac{10 \text{ rods}}{16 \text{ lbs.}}$. The definition of ratio is violated, hence the expression is not a proportion.

**276.** If from any proportion a new proportion is obtained by taking the antecedents for one ratio, and the consequents in the same order for the other ratio, the second proportion is deduced from the first by **alternation.**

$\frac{M}{N} = \frac{R}{S}$ is deduced by alternation from $\frac{M}{R} = \frac{N}{S}$.

Theorem VI may now be stated in the more convenient form, *If four quantities are in proportion, they are in proportion by alternation.*

## Proposition VII.

**277. Theorem.** *If four numbers or quantities are in proportion, the ratio of the second to the first equals the ratio of the fourth to the third.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.**

*To prove* $\frac{B}{A} = \frac{D}{C}$.

*Suggestion 1.* Let $\frac{A}{B} = m$. Then $\frac{C}{D} = m$. Why?

*2.* From the equations in suggestion 1, find the values of $A$ and $C$.

*3.* Divide by $A$ and $C$, respectively, both members of the two equations obtained in suggestion 2. Compare the results and reduce.

Another Method.

Compare the reciprocals of $\frac{A}{B}$ and $\frac{C}{D}$.

Query: Are any limitations to be placed upon the application of Proposition VII?

**278.** If from any proportion a new proportion is formed by inverting the ratios, the second proportion is deduced from the first by **inversion.**

Theorem VII may now be stated in the more convenient form: "If four quantities are in proportion, they are in proportion by inversion."

## PROPOSITION VIII.

**279. Theorem.** *If four numbers or quantities are in proportion, the ratio of the first plus the second to the second equals the ratio of the third plus the fourth to the fourth.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.**

*To prove* $\frac{A+B}{B} = \frac{C+D}{D}$.

*Suggestion 1.* Let $\frac{A}{B} = m$. Then $\frac{C}{D} =$ what?

2. Find value of $\frac{A+B}{B}$; *i. e.*, divide $A + B$ by $B$, ($A \div B = m$. Sug. 1).

3. Find value of $\frac{C+D}{D}$. Sug. 1.

4. Compare answers to sugs. 2 and 3.

*Therefore* —

Or, add 1 to each member of $\frac{A}{B} = \frac{C}{D}$.

**280.** If from the proportion $\frac{A}{B} = \frac{C}{D}$, the proportion $\frac{A+B}{B} = \frac{C+D}{D}$ is obtained, the second is deduced from the first by **composition.**

Theorem VIII may now be stated in the more convenient form: "*If four quantities are in proportion they are in proportion by composition.*"

## PROPOSITION IX.

**281. Theorem.** *If four numbers or quantities are in proportion, the ratio of the first minus the second to the second equals the ratio of the third minus the fourth to the fourth.*

**Let $\frac{A}{B} = \frac{C}{D}$ be a given proportion.**

*To prove that* $\frac{A-B}{B} = \frac{C-D}{D}$.

*Suggestion 1.* Let $\frac{A}{B} = m$. Then $\frac{C}{D} =$ what?

2. Find value of $\frac{A-B}{B}$; *i. e.*, divide $A-B$ by $B$.

3. What does $\frac{C-D}{D}$ equal?

4. Compare answers to suggestions 2 and 3.

*Therefore* —

Or, subtract 1 from each member of the equation $\frac{A}{B} = \frac{C}{D}$.

QUERY: May the terms of one ratio have a different unit of measure from the terms of the other ratio in Propositions VIII and IX? Why?

**282.** If from the proportion $\frac{A}{B} = \frac{C}{D}$, the proportion $\frac{A-B}{B} = \frac{C-D}{D}$ is obtained, the second is deduced from the first by **division.**

Theorem IX may now be stated in the more convenient form: If four quantities are in proportion, they are in proportion by **division.**

May the proportion $\frac{9^\circ}{6^\circ} = \frac{3\text{ ft.}}{2\text{ ft.}}$ be taken by composition or division? By inversion? By alternation?

283. A **continued proportion** is an equality of several ratios.

## Proposition X.

284. **Theorem.** *In a continued proportion, all of whose terms are numbers, or quantities of the same kind, the ratio of the sum of the antecedents to the sum of the consequents equals any of the ratios.*

**Let** $\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathbf{C}}{\mathbf{D}} = \frac{\mathbf{E}}{\mathbf{F}} = \frac{\mathbf{G}}{\mathbf{H}}$, **etc.**

*To prove* $\frac{A+C+E+G, \text{ etc.},}{B+D+F+H, \text{ etc.},} = \frac{A}{B}$ *or* $\frac{C}{D}$, *etc.*

*Suggestion 1.* Let each of the given ratios $= x$, and find the value of each antecedent.

*2.* Find the value of the sum of all the antecedents.

*3.* From the result of suggestion 2, find the value of $x$.

*4.* Compare the value of $x$, just found, with any of the given ratios.

*Therefore* —

---

Ex. 161. If a circle is inscribed in a right triangle the sum of the two legs of the triangle exceeds the hypotenuse by an amount equal to the diameter of the circle.

## Proposition XI.

285. **Theorem.** *The squares of the terms of a proportion are in proportion.*

Let $\frac{A}{B} = \frac{C}{D}$.

*To prove that* $\frac{A^2}{B^2} = \frac{C^2}{D^2}$.

286. Corollary I.—*Like powers of the terms of a proportion are in proportion.*

287. Corollary II.—*Like roots of the terms of a proportion are in proportion.*

---

Ex. 162. If $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A}{A-B} = \frac{C}{C-D}$, and also that $\frac{A-B}{A} = \frac{C-D}{C}$.

In deducing the second proportion from the first, are $A$, $B$, $C$, and $D$ restricted to being numbers, or may they be quantities as well; and, if quantities, are they all quantities of the same kind, or may they differ in kind?

Ex. 163. The circle inscribed in an equilateral triangle has the same center as the circle circumscribed about the triangle, and the radius of the circumscribed circle is double that of the inscribed circle.

Ex. 164. If an isosceles triangle is inscribed in a circle prove that the bisector of the vertical angle passes through the center of the circle.

## PROPOSITION XII.

**288. Theorem.** *If three terms of one proportion are respectively equal to the corresponding terms of another proportion, the fourth terms are equal.*

**Let** $\frac{A}{B}=\frac{C}{D}$, **and** $\frac{A}{B}=\frac{M}{D}$.

*To prove that* $C=M$.

*Suggestion 1.* Compare $\frac{C}{D}$ and $\frac{M}{D}$.

*2.* Compare $C$ and $M$.

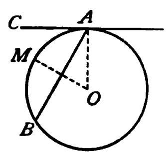
*Therefore* —

NOTE.—In all the preceding propositions in the theory of proportion, the theorems are true as stated, if the terms of the proportions are numbers.

If the terms are geometric magnitudes, or quantities, the limitations should be carefully studied and applied. Read § 198–202 (1) on measurement, and note their bearing on the theory of proportion.

---

Ex. 165. Prove that an angle formed by a tangent and a chord, is measured by one-half the intercepted arc, using the following construction: Drop a ⊥ from the center of ⊙ $O$ to the chord and extend to the arc, as at $M$. Connect $O$ and $A$.



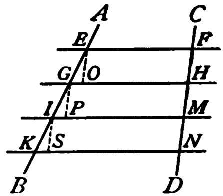

*Suggestion:* Compare $\angle MOA$ with $\angle BAC$.

Ex. 166. The median of a trapezoid bisects the diagonals, and any line whose extremities are in the bases of the trapezoid.

Ex. 167. Take the proportion $\frac{m}{n}=\frac{o}{s}$ by composition; by inversion; by alternation; by division; by inversion and composition.

## Proposition XIII.

**289. Theorem.** *If parallel lines intercept equal segments on one transversal, they intercept equal segments on all transversals.*




**Let E F, G H, etc., represent ∥ lines intercepting equal segments E G, G I, etc., on the transversal A B, and the segments F H, H M, etc., on the transversal C D.**

*To prove F H equals H M, etc.*

*Suggestion 1.* From $E$, $G$, etc., draw lines $EO$, $GP$, etc., ∥ to $CD$.

*2.* Compare $EO$ and $FH$, $GP$ and $HM$, etc. Give auth.

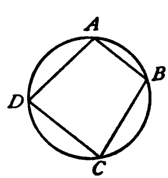
*3.* Compare $EO$, $GP$, etc.; $EO$ and $FH$; $GP$ and $HM$, etc. Give auth.

*4* Compare $FH$, $HM$, etc.

*Therefore* —

---

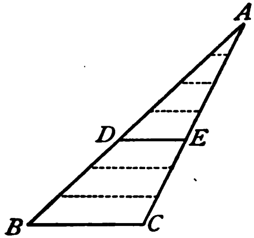
Ex. 168. If a quadrilateral is inscribed in a circle, the sum of the opposite angles equals two right angles. Prove that $\angle A$ plus $\angle C$ equals two right $\angle$s.




Ex. 169. A circle is described on one of the sides of an equilateral triangle as a diameter. Prove that the circumference bisects each of the other two sides.

## PROPOSITION XIV.

**290. Theorem.** *If a line is parallel to the base of a triangle, the ratio of the segments of one side equals the ratio of the segments of the other side.*




**Let D E be a line parallel to the base B C of the triangle A B C.**

To prove $\frac{AD}{DB} = \frac{AE}{EC}$.

CASE I. *When A D and D B are commensurable.*

*Suggestion 1.* Measure $AD$ and $DB$ by a common unit. What is the ratio of $AD$ to $DB$?

2. Through the points of division in $AD$ and $DB$ draw lines ∥ to $BC$ cutting $AC$.

3. Compare the size of the segments in $AE$ and $EC$. § 289.

4. Compare the number of segments in $AE$ and $EC$ with those in $AD$ and $DB$, respectively.

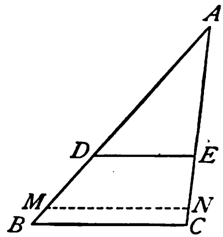
5. What is the ratio of $AE$ to $EC$? § 202.

6. Compare $\frac{AD}{DB}$ and $\frac{AE}{EC}$. Give auth.

*Therefore—*

Apply carefully the definition of ratio in Suggestion 5.

CASE II. *When A D and D B are incommensurable.*




*Suggestion 1.* Divide $A\,D$ into any number of equal parts, and lay off the unit of measure upon $D\,B$ as many times as possible. There will be a remainder, $M\,B$, less than the unit of measure. Why?

2. Through $M$ draw $M\,N \parallel$ to $B\,C$.

3. Compare $\frac{D\,M}{A\,D}$ with $\frac{E\,N}{A\,E}$. Give auth.

4. If the unit of measure be continually diminished, the ratio $\frac{D\,M}{A\,D}$ is a variable. Why?

5. What is its limit? Why? § 214.

6. The ratio $\frac{E\,N}{A\,E}$ is also a variable. What is its limit?

7. As the unit of measure decreases, how do the ratios $\frac{D\,M}{A\,D}$ and $\frac{E\,N}{A\,E}$ always compare? Sug. 3.

8. How do the ratios $\frac{D\,B}{D\,A}$ and $\frac{E\,C}{E\,A}$ compare? § 213.

*Therefore*—

NOTE.—Compare this demonstration with that of § 215.

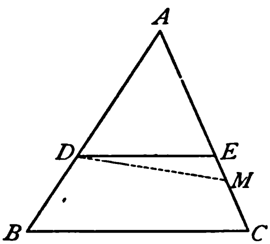
**291.** COROLLARY. *If a line be drawn parallel to the base of a triangle, one side is to either of its segments as the other side is to its corresponding segment.*

To prove $\frac{A\,B}{A\,D} = \frac{A\,C}{A\,E}$ or $\frac{A\,B}{D\,B} = \frac{A\,C}{E\,C}$.

*Suggestion.* Compare these proportions with the proportion in the conclusion of the proposition, and see by what authority in the theory of proportion the former can be deduced from the latter.

## PROPOSITION XV.

292. **Theorem.** *If a line divides two sides of a triangle proportionally, it is parallel to the base.*




**Let A B C be a triangle, and let D E divide the sides, so that** $\frac{AD}{DB} = \frac{AE}{EC}$.

*To prove that D E is parallel to B C.*

*Suggestion 1.* Take the proportion of the hypothesis by composition.

*2.* From *D* draw *D M* ∥ to *B C*, and compare $\frac{AB}{DB}$ with $\frac{AC}{MC}$. § 291.

*3.* In the proportions of Suggestions 1 and 2, compare *E C* and *M C*. § 288.

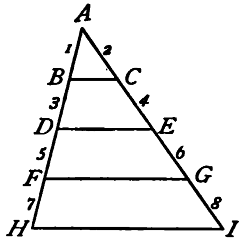
Complete the demonstration.

*Therefore—*

ANOTHER METHOD. Take the hypothesis $\frac{AD}{AB} = \frac{AE}{AC}$ and work out a demonstration.

## PROPOSITION XVI.

**293. Theorem.** *If several lines are drawn parallel to the base of a triangle intersecting the sides, the corresponding segments of the sides form a continued proportion.*




**Let AHI represent a triangle, and BC, DE, etc., lines ‖ to the base and intercepting the segments 1, 3, 2, 4, etc., on the sides AH and AI, respectively.**

*To prove* $\frac{1}{2} = \frac{3}{4} = \frac{5}{6}$, *etc.*

*Suggestion 1.* Compare $\frac{1}{2}$ with $\frac{3}{4}$. § 290 and § 274.

2. Compare $\frac{3}{4}$ with $\frac{AD}{AE}$; (§ 291). $\frac{5}{6}$ with $\frac{AD}{AE}$;(§ 274 and § 290;) $\frac{3}{4}$ with $\frac{5}{6}$ (Axiom 1).

3. Compare $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$, and extend the series.

*Therefore*—

---

Ex. 170. What has been done to the proportion $\frac{m}{n} = \frac{o}{s}$ to produce $\frac{m+o}{o} = \frac{n+s}{s}$? $\frac{m+n}{m} = \frac{o+s}{o}$? $\frac{m+n}{o+s} = \frac{n}{s}$?

## Similar Polygons.

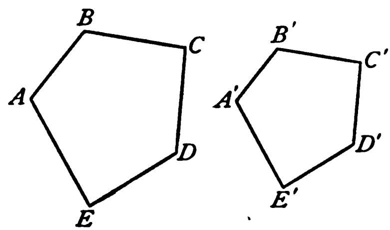
**294. Similar Polygons** are polygons which are mutually equiangular and have their corresponding sides proportional.

**295.** Points, lines, and angles, of similar polygons which are similarly situated, are **homologous.**

**296.** The **ratio of similitude** of similar polygons is the ratio of any two homologous sides.

**297.** COROLLARY. *From the definition it follows that in similar polygons, corresponding angles are equal and homologous sides are proportional.*

The polygons $A\,B\,C$, etc., and $A'\,B'\,C'$, etc., are similar, if $\angle A = \angle A'$, $\angle B = \angle B'$, etc., and $\frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'} = \frac{C\,D}{C'\,D'}$, etc.




The ratio $\frac{A\,B}{A'\,B'}$ or $\frac{B\,C}{B'\,C'}$, etc., is the **ratio of similitude** of the polygons.

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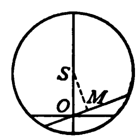
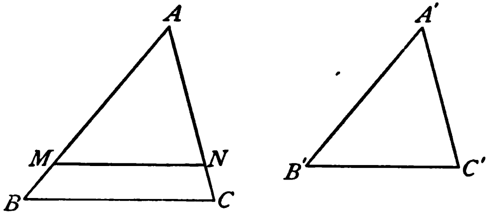
Ex. 171. The locus of the middle points of all chords which pass through a given point, is a circle whose diameter is the line connecting the given point and the center of the circle.




Prove that the circle described on $O\,S$ is the required locus.

## PROPOSITION XVII.

**298. Theorem.** *Two triangles which are mutually equiangular are similar.*




**Let ABC and A′B′C′ be two triangles in which the angle A is equal to the angle A′, the angle B is equal to the angle B′, and the angle C is equal to the angle C′.**

*To prove that the triangles $A\ B\ C$* and *$A'\ B'\ C'$* are *similar.*

*Suggestion 1.* What part of the definition of similar triangles remains to be proved? § 294.

*2.* Place $\triangle\ A'\ B'\ C'$ upon $\triangle\ A\ B\ C$, so that $A'\ B'$ lies upon $A\ B$, with $A'$ upon $A$, and $B'$ upon $A\ B$, or $A\ B$ extended, as at $M$.

*3.* What direction does $A'\ C'$ take? Why? Let $C'$ fall at some point, as at $N$.

*4.* $M\ N$ is $\parallel$ $B\ C$. Why?

*5.* $\dfrac{A\ B}{A\ M} = \dfrac{A\ C}{A\ N}$. (§ 291.) Hence $\dfrac{A\ B}{A'\ B'} = \dfrac{A\ C}{A'\ C'}$.

*6.* What remains yet to be proved?

*7.* Again place $\triangle\ B'\ A'\ C'$ upon $\triangle\ B\ A\ C$, $B'$ upon $B$, $B'\ A'$ upon $B\ A$, etc. Compare the ratios $\dfrac{A\ B}{A'\ B'}$ with $\dfrac{B\ C}{B'\ C'}$.

*8.* Compare $\dfrac{A\ B}{A'\ B'}$, $\dfrac{B\ C}{B'\ C'}$, and $\dfrac{A\ C}{A'\ C'}$. Give auth.

*Therefore—*

**299.** COROLLARY I. *If two triangles have two angles of one equal, respectively, to two angles of the other, the triangles are similar.*

**300.** COROLLARY II. *Two right triangles which have an acute angle of one equal to an acute angle of the other, are similar.*

**301.** SCHOLIUM. In similar polygons homologous sides are opposite equal angles, and equal angles are opposite homologous sides.

---

Ex. 172. Through one of the points common to two intersecting circumferences, draw the diameters of the circles and prove that the line connecting the other extremities of the diameters passes through the other point of intersection of the circumferences.

Ex. 173. The bisectors of all angles inscribed in the same segment pass through a common point.

Ex. 174. Draw any two equal chords of a circle that do not intersect. Connect their adjacent extremities. Prove the connecting chords are parallel.

Ex. 175. Draw any two parallel chords, connect their extremities. Prove the connecting chords are equal.

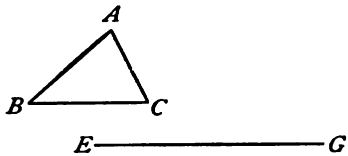
Ex. 176. Draw a tangent to a given circle that shall make a given angle with a given line.

Ex. 177. Given a triangle *A B C*, how could you use § 299 to construct a triangle similar to *A B C* upon a given line *M* homologous to *A C* of the triangle? Homologous to *A B* of the triangle?

Ex. 178. If triangle *A B C*, Ex. 177, is a right triangle, the right angle at *C*, solve the Exercise. § 300.

## PROPOSITION XVIII.

**302. Problem.** *To construct upon a given line, a triangle similar to a given triangle.*



**Let A B C be a given triangle and E G a given line.**

*To construct upon E G, a triangle similar to triangle A B C, E G being homologous to B C.*

Let X be the required vertex.

*Suggestion 1.* What must be true that the required △ $EGX$ may be similar to $ABC$? § 299.

2. Make the required construction.

---

Ex. 179. If you take a pole as long as you are tall, lie upon your back with the pole upright at your feet, and sight over the top of the pole to some adjacent object, how high is the object?

What measurement must you take to find out?

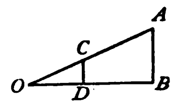
Ex. 180. How could you measure the height of a church spire, a tree or a building? How could you use shadows to find heights of objects?

Ex. 181. Determine as accurately as possible the height of the gable of your school building.

NOTE.—Not over two should work together, and answers should be compared in class.

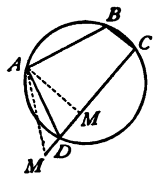
Ex. 182. To measure height *A B*.

*Suggestion.* Set up a pole perpendicularly at a convenient point, as at *D*. From *O* sight to *A*. Let another person move a card upon the pole until a point in line *A O* is determined on the pole. Make the required measurements and determine the height *A B*. § 298.



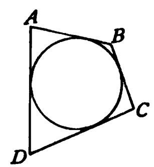

NOTE.—The line O B may be the line of the ground or some other convenient line.

Ex. 183. If the sum of the opposite angles of a quadrilateral equals two right angles, prove that a circumference is possible through the four vertices.

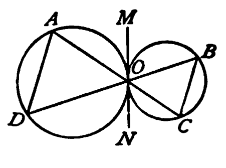



*Suggestion.* A circumference can be described through three vertices, as *A*, *B* and *C*. If it does not pass through *D*, it must cut the line *C D*, or *C D* extended, as at *M*. Compare the sum of ∠s *B* and *C M A* with two right ∠s. Also, the sum of ∠s *B* and *C D A* with two right ∠s.

Ex. 184. If a quadrilateral is circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.




Ex. 185. If two circles are tangent, and two secants are drawn through the point of contact, the chords joining the intersections of the secants and the circumferences are parallel.

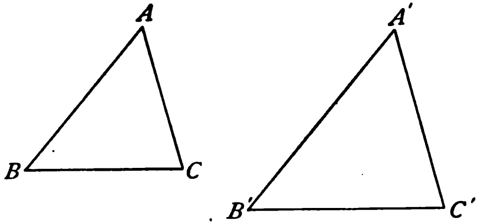



Prove *A D* parallel to *B C*.

*Suggestion.* Draw the common tangent, *M N*. Compare ∠ *A O M* with ∠ *C O N*. Compare ∠ *A O M* with ∠ *A D O*; also, ∠ *C O N* with ∠ *O B C*.

## Proposition XIX.

303. **Theorem.** *If two triangles have an angle of one equal to an angle of the other, and the sides including the equal angles proportional, the triangles are similar.*




**In the triangles A B C and A′ B′ C′, let the angle A equal the angle A′ and let the ratio $\frac{AB}{A'B'}$ equal the ratio $\frac{AC}{A'C'}$.**

*To prove that the triangles A B C and A′ B′ C′ are similar.*

*Suggestion 1.* What must be proved in addition to the hypothesis to make the triangles similar according to the definition?

*2.* Place the △ *A B C* upon the △ *A′ B′ C′*, with *A* upon *A′*, and *A B* and *A C* upon *A′ B′* and *A′ C′*, respectively. Why is this possible?

*3.* Where do *B* and *C* fall?

*4.* *B C* is ‖ to *B′ C′*. Why? § 292.

*5.* How do the ∠s *B* and *C* compare with the ∠s *B′* and *C′*, respectively? Why?

*6.* Complete the demonstration.

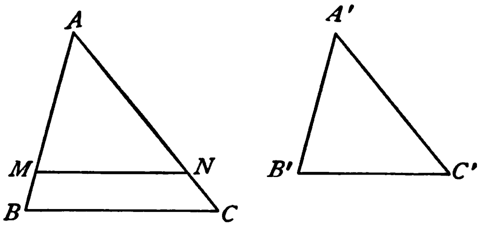
*Therefore—*

---

Ex. 186. Let *A B C* and *A′ B′ C′* be two similar triangles. *A B* = 7 feet, *A′ B′* = 10 feet, *A C* = 14 feet. Find the length of *A′ C′*. If *A B* is 13 feet, *A′ B′* and *A C* 10 and 11 feet, respectively, find the length of *A′ C′*.

## PROPOSITION XX.

**304. Theorem.** *Two triangles whose corresponding sides are proportional are similar.*




**Let ABC and A′B′C′ be two triangles such that** $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$.

*To prove that triangles A B C and A′ B′ C′ are similar.*

*Suggestion 1.* What remains to be proved according to the definition of similar triangles?

*2.* Upon $A\ B$ lay off $A\ M$ equal to $A'\ B'$, and upon $A\ C$ lay off $A\ N$ equal to $A'\ C'$; connect $M$ and $N$. Compare △s $A\ M\ N$ and $A\ B\ C$. § 303.

*3.* Compare the ratios $\frac{A\ M}{A\ B}$ with $\frac{M\ N}{B\ C}$; also $\frac{A'\ B'}{A\ B}$ and $\frac{B'\ C'}{B\ C}$.

*4.* Compare $M\ N$ and $B'\ C'$; give auth. (§ 288); △s $A\ M\ N$ and $A'\ B'\ C'$. Give auth.

*5.* Compare △ $A\ B\ C$ and △ $A'\ B'\ C'$.

*Therefore—*

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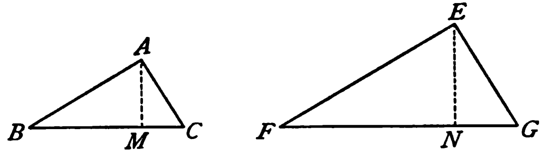
Ex. 187. The sides of triangle $A\ B\ C$ are, respectively, 4, 8, and 11 feet. In a similar triangle, $A'\ B'\ C'$, the side homologous to 4, is 6 feet. Find the other two sides of triangle $A'\ B'\ C'$.

Give a summary of the tests for similarity of triangles.

## PROPOSITION XXI.

305. **Theorem.** *The ratio of homologous altitudes of similar triangles is equal to the ratio of similitude of the triangles.*




**Let A M and E N be homologous altitudes in the similar triangles A B C and E F G, and let $\frac{AB}{EF}$ represent the ratio of similitude of the triangles.**

*To prove that* $\frac{AM}{EN} = \frac{AB}{EF}$.

*Suggestion 1.* Compare $\triangle$s $AMB$ and $ENF$. § 300. Complete the demonstration.

306. COROLLARY. *In similar triangles homologous altitudes are proportional to the bases.*

*To prove* $\frac{AM}{EN} = \frac{BC}{FG}$.

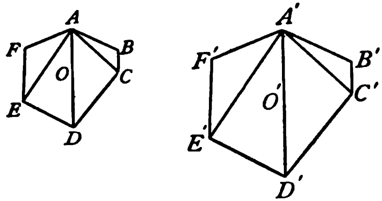
---

Ex. 188. Through a given point, to draw a straight line so that the portion of it intercepted between two given intersecting lines is divided at the point into two equal parts.

*Suggestion.* Through the point draw a line parallel to one of the given lines. § 290.

## PROPOSITION XXII.

**307. Theorem.** *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*




**Let the polygons ABC, etc., and A′B′C′, etc., be composed of the same number of triangles, similar each to each and similarly placed, ABC being similar to A′B′C′, etc.**

*To prove that the polygons are similar.*

*Suggestion 1.* What must be proved to know that the polygons are similar? § 294.

2. Compare $\angle B$ with $\angle B'$, $\angle C$ with $\angle C'$, $\angle D$ with $\angle D'$, etc. Give auth.

3. Compare the ratio $\dfrac{AB}{A'B'}$ with $\dfrac{BC}{B'C'}$. Give auth.

4. Compare the ratio $\dfrac{BC}{B'C'}$ with $\dfrac{AC}{A'C'}$; also the ratio $\dfrac{CD}{C'D'}$ with $\dfrac{AC.}{A'C'}$. How, then, does $\dfrac{BC}{B'C'}$ compare with $\dfrac{CD}{C'D'}$?

5. In a similar manner compare the ratios of other corresponding sides in the two polygons.

*6.* Apply the definition of similar polygons. Sug. 1.

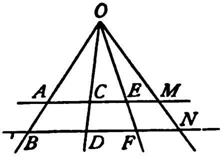
*Therefore—*

**308.** In this early study of the subject, in stating a proportion deduced from two similar polygons, homologous sides should be selected for the terms of each ratio; the antecedents of the ratios should be taken from one of the polygons and the consequents from the other.

If any other form of the proportion is desired, it can be modified by the appropriate proposition in the theory of proportion. For instance, if triangle I is similar to triangle II, and $A$ is homologous to $A'$, etc., the proportion $\frac{A}{A'} = \frac{B}{B'}$ may be taken. (Apply above test.) If the proportion, $\frac{A}{B} = \frac{A'}{B'}$, is desired, the original proportion can be taken by alternation. Later, when the pupil is sure of himself, the work can be abbreviated, and any desired proportion taken at once.

---

Ex. 189. Straight lines drawn through any point intercept proportional segments upon two parallel lines.




Prove that $\frac{AC}{BD} = \frac{CE}{DF} = \frac{EM}{FN}$, etc.

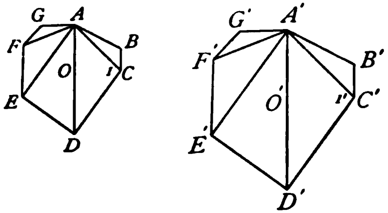
*Suggestion.* Compare each of the ratios $\frac{AC}{BD}$ and $\frac{CE}{DF}$ with the ratio $\frac{OC}{OD}$. Complete the demonstration.

Ex. 190. Upon a given base construct a right triangle, having given the perpendicular from the right angle to the hypotenuse.

## PROPOSITION XXIII.

**309. Theorem.** CONVERSE OF PROPOSITION XXII.

*Two similar polygons can be divided into the same number of triangles, similar each to each and similarly placed.*




**Let polygons ABC, etc., and A′B′C′, etc., be similar, and let all possible diagonals be drawn from two corresponding vertices, as A and A′.**

*To prove that the triangles in one polygon are similar, respectively, to those in the other polygon.*

*Suggestion 1.* What must be proved to establish the theorem? § 298, § 303 or § 304.

*2.* Compare $\triangle\ A\ B\ C$ and $\triangle\ A'\ B'\ C'$.

*3.* Compare the ratios $\frac{BC}{B'C'}$ with $\frac{AC}{A'C'}$; $\frac{BC}{B'C'}$ with $\frac{CD}{C'D'}$. Give auth.

Then, how do $\frac{AC}{A'C'}$ and $\frac{CD}{C'D'}$ compare? Give auth.

*4.* Compare $\angle\ 1$ and $\angle\ 1'$.

*5.* Compare $\triangle$s $A\ D\ C$ and $A'\ D'\ C'$. Give auth.

*6.* In a similar manner, compare the other pairs of $\triangle$s. Complete the demonstration.

*Therefore—*

NOTE.—It would be profitable to compare the demonstrations in § 307 and § 309.

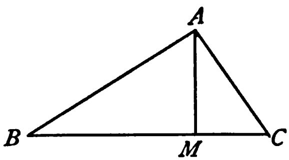
## PROPOSITION XXIV.

**310. Theorem.** *If a perpendicular be dropped to the hypotenuse from the vertex of the right angle in a right triangle:*

PART I. *The triangles thus formed are similar to each other and to the whole triangle.*

PART II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

PART III. *Each side is a mean proportional between the hypotenuse and the segment adjacent to that side.*




**Let A B C represent a right triangle, the right angle at A, and A M the perpendicular drawn from A to the hypotenuse B C.**

PART I. To prove that the triangles $A\ M\ C$, $A\ M\ B$, and $B\ A\ C$ are similar.

*Suggestion.* What are the tests for similarity of $\triangle$s? (§§ 298, 303, and 304.) Select the one suited to this case.

PART II. *To prove* $\frac{BM}{AM} = \frac{AM}{MC}$. (See Ex. 119.)

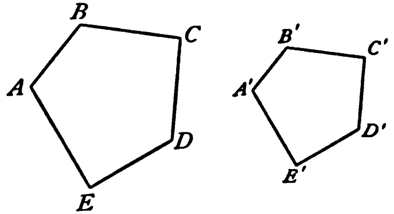
*Suggestion.* Select two $\triangle$s, one of which contains the two antecedents and the other the two consequents, and determine the required proportion. § 308.

PART III. *To prove that* $\frac{CB}{AB} = \frac{AB}{BM}$; or, *that* $\frac{CB}{CA} = \frac{CA}{CM}$.

*Suggestion.* See Sug. Part II.

### PROPOSITION XXV.

311. **Theorem.** *The ratio of the perimeters of two similar polygons equals the ratio of similitude of the polygons.*




**Let polygons A D and A′D′ be similar.**

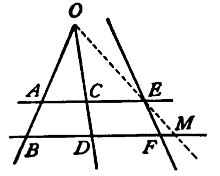
*To prove that the ratio,* $\frac{\textit{perimeter } A\ B\ C, \textit{ etc.},}{\textit{perimeter } A'\ B'\ C', \textit{ etc.},}$, *is equal to the ratio of similitude* $\frac{A\ B}{A'\ B'}$, *or* $\frac{B\ C}{B'\ C'}$, *etc.*

*Suggestion.* § 284.

*Therefore*—

---

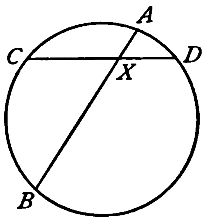
Ex. 187A. If two parallel lines are cut proportionally by a set of secant lines, prove that the secant lines pass through a common point. If $\frac{A\ C}{B\ D} = \frac{C\ E}{D\ F}$, etc., prove that the lines $B\ A$, $D\ C$, $F\ E$, etc., intersect at a common point.




*Suggestion.* Extend two of them, as $B\ A$ and $D\ C$, until they meet at $O$. Connect $O$ and $E$, and extend $O\ E$ to the other of the parallels at $M$. Compare $E\ M$ and $E\ F$.

## PROPOSITION XXVI.

**312. Theorem.** *If two chords of a circle intersect, the ratio of either segment of the first to either segment of the second is equal to the ratio of the remaining segment of the second to the remaining segment of the first.*




**Let the chords AB and CD intersect at X.**

*To prove that* $\frac{AX}{DX} = \frac{CX}{BX}$, *or* $\frac{AX}{CX} = \frac{DX}{BX}$.

*Suggestion 1.* As no △s are given in the theorem, two △s must be constructed that contain, one the required antecedents, and the other the required consequents. Auth.

*2.* Prove the constructed △s are similar.

*3.* Establish the required proportion.

*Therefore—*

Establish both proportions in the special enunciation.

---

Ex. 188A. A perpendicular dropped from the circumference of a circle upon the diameter is a mean proportional between the segments of the diameter.

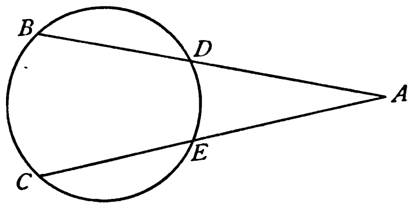
Ex. 189A. Use Ex. 188A to construct a mean proportional to the lines *A* and *B*.

*A*

*B*

## PROPOSITION XXVII.

**313. Theorem.** *If two secants intersect without a circle, the ratio of the first secant to the second is equal to the ratio of the external segment of the second to the external segment of the first.*




**Let A B and A C represent two secants meeting at A and intersecting the circle in B, D, E, and C.**

*To prove that* $\frac{AB}{AC} = \frac{AE}{AD}$.

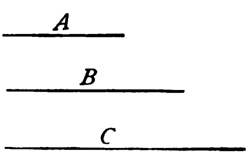
*Suggestion.* Make two △s, by drawing construction lines, according to the principle laid down in Suggestion 1, § 312.

Complete the demonstration.

---

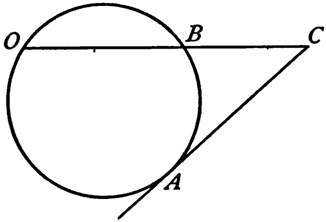
EX. 190A. By § 312 find a line $X$, that is a fourth proportional to $A$, $B$ and $C$; *i. e.*, if $\frac{A}{B} = \frac{C}{X}$, find $X$. § 259.

*Suggestion.* If the lines $A$, $B$, and $C$ are located as suggested by the theorem, three points are fixed by which the circle can be constructed. Then $X$ can be found.



## Proposition XXVIII.

**314. Theorem.** *If a secant and a tangent meet without a circle, the tangent is a mean proportional between the secant and its external segment.*




**Let AC represent a tangent, and OC a secant meeting the tangent at the point C.**

*To prove that* $\frac{OC}{AC} = \frac{AC}{BC}$.

*Suggestion.* Construct △s, so that the antecedents are in one △ and the consequents in another.

By the method suggested in § 312, complete the demonstration.

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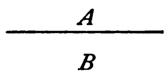
Ex. 191. A mean proportional between two lines can be constructed by this theorem, § 314. Try it.

$\frac{A}{X} = \frac{X}{B}$. Find $X$.



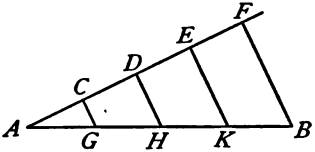

Ex. 192. Construct a fourth proportional to three given lines by § 313. (Sug. for Ex. 190A.)

Ex. 193. Construct a third proportional to two lines. § 310, II or III.

Ex. 194. Construct a mean proportional to $A$ and $B$ by § 310, Part II. By Part III.

## PROPOSITION XXIX.

**315. Problem.** *To divide a given straight line into any number of equal parts.*




**Let A B be the given straight line.**

*To divide A B into any number of equal parts; for example, into four parts.*

*Suggestion.* In what propositions has the truth that a line is divided into equal parts been established?

§ 289, § 293, etc.

Make the construction in accordance with the truth selected.

*Further suggestions:* *1.* Draw an indefinite line through *A*, making any convenient ∠ with the given line.

*2.* Lay off upon this line from *A* an estimated fourth of *A B*, four times. Connect the last point of division with *B*. Complete the solution.

---

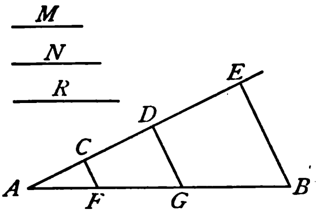
Ex. 195. What is the locus of the vertex of the right angle of a right-angled triangle constructed upon a given line as a hypotenuse? (See § 222.)

Ex. 196. Construct upon a given line, as a chord, a segment of a circle which can contain a given angle.

*Suggestion.* Construct at the ends of the given line, and upon it, angles whose sum is the supplement of the given angle. Circumscribe a ⊙ about the △ thus formed.

### PROPOSITION XXX.

**316. Problem.** *To divide a given straight line into parts proportional to any given lines.*




**Let A B be the given straight line, and M, N, R, etc., be the given lines.**

*To divide A B into parts such that the first part is to M as the second part is to N, as the third part is to R, etc.*

*Suggestion 1.* In what proposition has the truth concerning the division of a line into several proportional parts been demonstrated? § 293.

Apply the truth in the solution of the problem. (See method, § 315.)

---

Ex. 197. With a given fixed base, and a given vertical angle of a triangle, find the locus of the vertex. (See Exercise 196.)

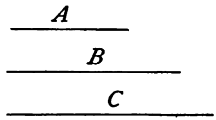
Ex. 198. Construct a triangle whose base and vertical angle are given, and whose vertex is at a given distance from the middle point of the base.

*Suggestion.* Solve by intersection of two loci.

Ex. 199. If two circles intersect and their common chord be extended, prove that tangents drawn to the two circles from any point in the chord extended are equal.

### PROPOSITION XXXI.

**317. Problem.** *To construct a fourth proportional to three given lines.*




**Let A, B, and C be three given lines.**

*To construct a fourth proportional to A, B, and C.*

*Suggestion.* Employ one of the many propositions from which a proportion of four different terms has been determined. § 290, § 291, § 298, § 303, § 304, § 312, Ex. 189, etc.

Make several solutions.

Illustration of solution, using Exercise 189.

Let $A$, $B$, and $C$ be the given lines.

To find $X$, the fourth proportional to $A$, $B$, and $C$.

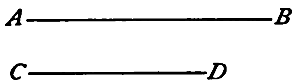
In Exercise 189, $\frac{A\,C}{B\,D} = \frac{C\,E}{D\,F}$. As our proportion is to read $\frac{A}{B} = \frac{C}{X}$, $A$, $B$, $C$, and $X$ must take the relative positions in the figure occupied, respectively, by $A\,C$, $B\,D$, $C\,E$, and $D\,F$.

Since $A\,C$ and $B\,D$ are ‖, draw two indefinite ‖ lines, and lay off $A$ and $B$ in the same relative positions as $A\,C$ and $B\,D$. $O\,B$ and $O\,D$ of the figure are determined by the ends of $A\,C$ and $B\,D$. Hence, draw lines through ends of $A$ and $B$ and extend until they meet, as at $O$. They must meet unless $A$ and $B$ are equal. Why? § 151.

$C$ can now be laid off in position of $C\,E$, an extension of $A$. The line corresponding to $O\,E$ is now determined by points $O$ and the end of $C$. Hence, draw a line through $O$ and end of $C$, meeting line $B$ extended. $X$ is now determined.

## Proposition XXXII.

**318. Problem.** *To construct a third proportional to two given lines.*




**Let A B and C D be two given lines.**

*To construct a third proportional to A B and C D.*

Query. Can you find a third proportional by means of § 290, § 312, § 313, Exercise 189, etc.? Try it.

*Suggestion.* Every proposition in which a mean proportional has been established furnishes the basis of a solution also. § 310, II, § 310, III, § 314.

Name all the theorems on which a solution can be based.

---

Ex. 200. What is the locus of the vertex on one side of the base of a triangle having a given base and a given altitude?

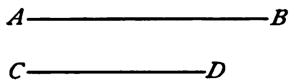
Ex. 201. What is the locus of the vertex of a triangle having a given base and a given altitude?

Ex. 202. Construct a triangle having a given base, a given altitude, and a given vertical angle. Ex. 197.

*Suggestion.* Solve by intersection of two loci.

## Proposition XXXIII.

**319. Problem.** *To construct a mean proportional to two given lines.*




**Let A B and C D be two given lines.**

*To construct a mean proportional to A B and C D.*

A solution can be based on § 310, II, 310, III, and § 314.

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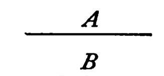
Ex. 203. Prove Proposition XXI, § 305, by taking $\frac{BC}{FG}$ as the ratio of similitude of the triangles.

Ex. 204. Construct a line that shall equal the $\sqrt{ab}$.




*Suggestion 1.* To find line $x$.

Then $x = \sqrt{ab}$ (§ 268 and § 268-note) (2) $xx = ab$; Why? (3) § 264, (4) § 319.

Ex. 205. Construct a line that shall equal $\sqrt{2}$. Suggestion. $\sqrt{2} = \sqrt{2 \times 1}$.

Select a line for a unit.

Ex. 206. Construct a line that shall equal $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$; $\sqrt{10}$; $\sqrt{12}$.

Make two solutions each for $\sqrt{6}$ and $\sqrt{10}$. Make three for $\sqrt{12}$. *Sug.* $\sqrt{6} = \sqrt{2 \times 3}$, or $\sqrt{6 \times 1}$.

Ex. 207. $AB$ is a chord of a circle. $CD$ is a diameter perpendicular to $AB$, and intersecting $AB$ at $E$. $CE$ is 5 in. and $AC$ 10 in. Find the diameter of the circle.

### 320. Review and Summary.

*1.* If $\frac{M}{N} = \frac{S}{O}$, name at sight what changes in the theory of proportion have taken place in the following, and verify your statement:

(*a*) $\frac{N}{M} = \frac{O}{S}$.

(*b*) $\frac{M}{S} = \frac{N}{O}$.

(*c*) $\frac{N}{M+N} = \frac{O}{S+O}$.

(*d*) $\frac{N}{O} = \frac{M+N}{S+O}$.

(*e*) $\frac{M+N}{M} = \frac{S+O}{S}$.

(*f*) $\frac{M+S}{S} = \frac{N+O}{O}$.

(*g*) $\frac{S}{M-S} = \frac{O}{N-O}$.

(*h*) $\frac{M-N}{M} = \frac{S-O}{S}$.

Extend the list.

*2.* Name all the cases in which a mean proportional has been found.

*3.* Name all the methods by which triangles are found to be similar.

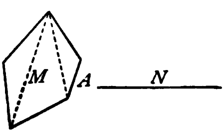
*4.* Name the methods by which two polygons can be found to be similar.

*5.* By what theorems can third proportionals be constructed?

*6.* By what theorems can mean proportionals be constructed?

---

Ex. 208. Construct a polygon similar to a given polygon upon a given line homologous to a given side of the polygon. Construct a polygon similar to $M$ upon line $N$ homologous to $A$. §§302, 307.



# CHAPTER IV.

## COMPARISON AND MEASUREMENT OF POLYGONS.

### Definition.

**321.** The **area** of a surface is its ratio to some selected unit of measure, times the unit of measure. §§ 197 to 202.

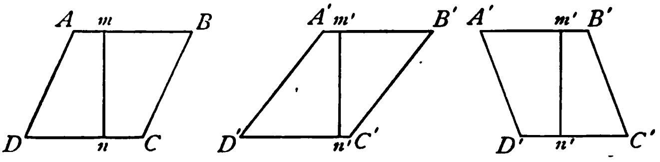
If $M$ denotes the unit of measure for determining the area of surface $A$, and if the ratio $\frac{A}{M}$ equals $b$, the area of the surface $A$ is $b\,M$; i. e., $b$ times the unit $M$.

**322.** The **unit of measure** for surfaces is a square, whose side is a given linear unit; as a square inch, a square foot, etc.

An acre is the one exception to this definition.

### Proposition I.

**323. Theorem.** *Two parallelograms having equal bases and equal altitudes, are equal in area.*




**Let A C and A′ C′ be two parallelograms having the bases D C and D′C′ equal, and the altitudes *m n* and *m′ n′* equal.**

*To prove that the area of A C is equal to the area of A′ C′.*

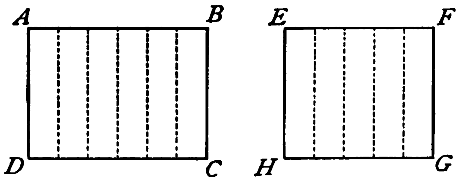
*Suggestion.* Place ▱ $A\,C$ upon ▱ $A'\,C'$, $D\,C$ upon $D'\,C'$, and compare the parts external to each other.

*Therefore* —

Make drawings for the superposition of one parallelogram upon the other in many different forms, but all under the conditions of the theorem.

## PROPOSITION II.

**324. Theorem.** *If two rectangles have equal altitudes, the ratio of their areas is equal to the ratio of their bases.*




**Let A C and E G be two rectangles having equal altitudes, D A and H E.**

*To prove that* $\frac{\text{area of } A\,C}{\text{area of } E\,G}$ *is equal to* $\frac{D\,C}{H\,G}$.

CASE I. *When D C and H G are commensurable.*

*Suggestion 1.* By thorough preparation on § 215 and § 290, this proposition can be demonstrated as an original exercise.

*2.* Measure $D\,C$ and $H\,G$ by some common unit of measure. Let the unit be contained $m$ times in $D\,C$, and $n$ times in $H\,G$.

*3.* What does the ratio of $D\,C$ to $H\,G$ equal? Why?

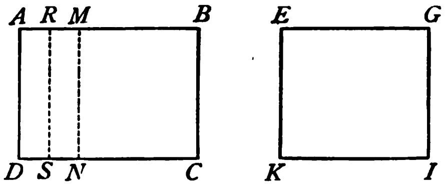
*4.* At the points of division erect ⊥s to $D\,C$ and $H\,G$, and extend to the secondary bases, thus dividing $A\,C$ and $E\,G$ into rectangles. Why rectangles?

*5*. How do these rectangles compare in size? Why? How many rectangles in *A C*? In *E G*? Why?

*6*. What does the ratio of the areas of the rectangles *A C* and *E G* equal? Why? § 202.

*7*. Compare the ratio of the areas with the ratio of the bases.

CASE II. *When the bases are incommensurable.*




*Suggestion 1*. Take any unit of *K I* and lay it off on *D C* as many times as possible. There must be a remainder less than one of these parts. Why? Suppose the unit of *K I* is contained in *D C* a certain number of times with a remainder *D N*. Erect *N M* ⊥ to *D C*, at the point *N*.

*2*. Compare the ratio of the areas of *M C* and *E I* with the ratio of *N C* and *K I*.

*3*. Apply a unit of measure smaller than *D N*. Let the remainder be *D S*. Erect a ⊥ as before. Compare ratio $\frac{RC}{EI}$ with $\frac{SC}{KI}$. Let the unit continually decrease.

*4*. The ratio $\frac{RC}{EI}$ is a variable. Why? What is its limit? What is the limit of variable $\frac{SC}{KI}$?

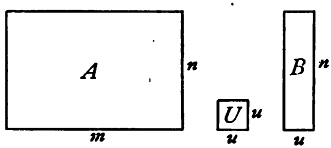
*5*. Compare the limits.

*Therefore* —

**325**. COROLLARY.—*If two rectangles have equal bases, the ratio of their areas is equal to the ratio of their altitudes.*

## PROPOSITION III.

**326. Theorem.** *The number of units of area in any rectangle is equal to the product of the measures of its base and altitude.*




**Let A represent a rectangle, U a unit of measure for area, and *u* the linear unit, viz., a side of the square U. Let *u* be contained *a* times in base *m*, and *b* times in altitude *n*; i. e.,** $\frac{m}{u} = a$ **and** $\frac{n}{u} = b$.

*To prove that* $\frac{A}{U}$ *is equal to* $a \times b$, *or that A contains the unit U* $a \times b$ *times.*

*Suggestion 1.* Construct a rectangle *B* whose altitude is equal to *n* and whose base is equal to *u*.

2. What is the ratio of *A* to *B*? Of *B* to *U*? § 324.

3. What, then, is the ratio of *A* to *U*? § 269.

*Therefore* —

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Ex. 209. The ratio of the squares of the legs of a right triangle is equal to the ratio of the segments of the hypotenuse formed by dropping a perpendicular from the vertex of the right angle upon the hypotenuse.

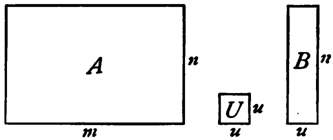
*Suggestion 1.* See figure in § 31c. By use of Part III of the proposition in § 310, find an expression for the square of each leg of the triangle. § 268.

2. What is the ratio of the squares found?

### Model.

PROPOSITION III.

**Theorem.** *The number of units of area in any rectangle is equal to the product of the measures of its base and altitude.*




**Let A represent a rectangle, U, a unit of measure for area, and *u* the linear unit, viz., a side of the square U. Let *u* be contained *a* times in base *m* and *b* times in altitude *n*; i, e., let $\frac{m}{u} = a$, and $\frac{n}{u} = b$.**

*To prove that* $\frac{A}{U} = a \times b$, *or that A contains the unit U* $a \times b$ *times.*

Construct rectangle *B* with base equal to *u* and altitude equal to *n*.

$\frac{A}{B} = \frac{m}{u} = a$. §324. $\frac{B}{U} = \frac{n}{u} = b$. §325. $\frac{A}{U} = a \times b$. §269.

Q. E. D.

327. COROLLARY.—*The area of a rectangle is equal to the product of the measures of the base and altitude times the unit of measure.*

Since $\frac{A}{U} = a \times b$, $A = (a \times b)\, U$, i. e., the area of *A* is $a \times b$ times the unit *U*.

*Illustration:* If a rectangle is 6 ft. by 10 ft., the area is $(6 \times 10)$ square feet.

**328.** Scholium I. In the applications of this theorem the base and altitude must be expressed in terms of the same unit, and the unit of area must be the square whose side is the linear unit.

**329.** Scholium II. The expression, "the product of the base and altitude," is a common abbreviation for "the product of the measures of the base and altitude times the unit of area."

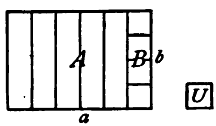
(*a*) "The product of two lines," or "the square of a line," when "square" means the algebraic second power, must not be interpreted in any other sense than that just stated. With this interpretation, the Corollary is usually stated as follows: The area of a rectangle is equal to the product of its base and altitude.

(*b*) The numbers which represent the measures of the lines may be integral, fractional, or incommensurable.

(*c*) The **rectangle of two lines** is an expression which is sometimes used instead of the product of two lines.

Note.—Possibly the thought in the foregoing demonstration can be made clearer by an illustration in the arithmetical form of analysis.

(a) Since $A$ contains $B$ $a$ times, and $B$ contains $U$ $b$ times, $A$ must contain $U$ $a \times b$ times.




(b) To take a particular case, let the base and altitude be measured by some linear unit, as 1 inch; and suppose this unit is contained 4 times in the altitude and 6 times in the base; then, as seen in the figure, there are 6 columns, with 4 squares in each column, and hence, in all, 6 x 4 squares; i. e., 24 square inches.

It can be readily seen, by constructing a figure, that if the unit is contained a fractional number of times in the base and altitude the same rule is reached.

Query: Can the conclusion of the proposition, in all its generality, be drawn from the method in common use in arithmetic? § 324. Case II

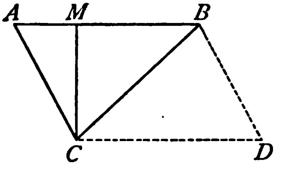
## PROPOSITION IV.

**330. Theorem.** *The area of a parallelogram is equal to the product of its base and altitude.*

*Suggestion.* § 323 and § 326.

## PROPOSITION V.

**331. Theorem.** *The area of a triangle is equal to one-half the product of its base and altitude.*




**Let A B C represent a triangle, A B its base, and M C its altitude.**

*To prove that the area of the triangle A B C is equal to one-half of the product of A B and M C.*

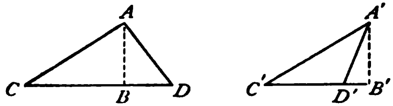
*Suggestion 1.* Enlarge △*A B C* so as to make a ▱ with *A B* and *A C* as two of its sides. State authority for your construction.

*2.* Compare the figures *A B C* and *A B D C* in respect to area. Compare their bases and altitudes.

*3.* What is the area of △ *A B C* in terms of its base and altitude?

**332.** COROLLARY I.—*A triangle is equal in area to one-half a rectangle if it has the same base and altitude.*

**333.** COROLLARY II.—*If two triangles have equal altitudes their areas have the same ratio as their bases.*




In triangles $A\,C\,D$ and $A'\,C'\,D'$, let the altitudes $A\,B$ and $A'\,B'$ be equal.

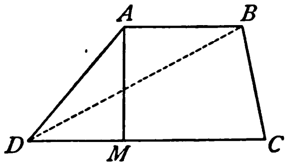
Then $\dfrac{A\,C\,D}{A'\,C'\,D'} = \dfrac{\frac{1}{2}\,A\,B \times C\,D}{\frac{1}{2}A'\,B' \times C'\,D'} = \dfrac{C\,D'}{C'\,D}$. Give auth. (§ 272.)

In a similar manner, prove that if two triangles have equal bases their areas have the same ratio as their altitudes.

334. Scholium.—For the interpretation of the expression, "the product of the base and altitude," in propositions IV and V, see § 329.

## Proposition VI.

335. **Theorem.** *The area of a trapezoid is equal to one-half the product of its altitude and the sum of its bases.*




**Let A B C D represent a trapezoid, and A M its altitude.**

*To prove that the area of the trapezoid, A B C D, is equal to one-half the product of A M and the sum of A B and D C.*

*Suggestion 1.* Draw the diagonal, $B\,D$.

*2.* Let $A\,B$ be the base of $\triangle\,D\,A\,B$, $D\,C$ be the base of $\triangle\,D\,C\,B$, and $A\,M$ the altitude of each. Why is $A\,M$ the altitude of each $\triangle$?

*3.* Find the area of each $\triangle$. Find the area of the trapezoid.

*Therefore* —

*Further suggestions:*

Area $DAB = \frac{1}{2}(AM \times AB)$.

Area $DBC = \frac{1}{2}(AM \times DC)$.

Area $DABC = \frac{1}{2}(AM \times AB) + \frac{1}{2}(AM \times DC) = \frac{1}{2} AM(AB + DC)$.

Explain the reductions algebraically.

QUERY: In terms of what other lines can the area of a trapezoid be expressed? Ex. 139.

## Area of a Polygon.

**336.** Various methods have been used to find the area of irregular polygons. Among the methods used the following may be noticed:

From any vertex of the polygon draw all possible diagonals, as in Fig. 1. The polygon is by this means divided into triangles, and if the bases and altitudes of these triangles are measured, their areas can be computed, and by addition the area of the polygon can be found.

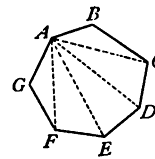


FIG. 1.

Another method is to draw the longest diagonal of the polygon, and from the vertices drop perpendiculars upon this diagonal, as in Fig. 2. The polygon is, in this way, divided into triangles and trapezoids, and if the bases and altitudes of these triangles and trapezoids are measured, their areas can be computed and the area of the polygon obtained.

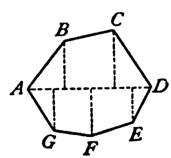


FIG. 2.

Still another method is to draw, through any vertex of the polygon, a straigth line, exterior to the polygon, and from the vertices drop perpendiculars upon this line, as in Fig. 3. In this way, triangles and trapezoids are formed. If the various bases and altitudes are measured, the areas of the triangles and trapezoids can be computed, and if the areas of the parts exterior to the polygon be subtracted from the sum of the other areas, the difference is the area of the polygon.

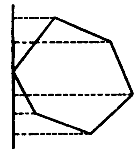


FIG. 3.

The method just described is the one by which surveyors sometimes compute irregular areas bounded by straight lines. If the map or outline of a field, or any irregular polygon, is drawn to scale, any line of the polygon can be determined by means of the homologous line in the constructed polygon. § 294.

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Ex. 210. The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.

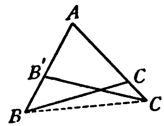
*Suggestion 1.* See Sug. 1, of Ex. 209.

*2.* Add the squares.

Ex. 211. To construct a triangle similar to a given triangle having a given perimeter.

Ex. 212. If two triangles have an angle of one equal to an angle of the other, the ratio of their areas equals the ratio of the products of the sides including the equal angles.

*To prove* $\frac{\triangle ABC}{\triangle AB'C'} = \frac{AB \times AC}{AB' \times AC'}$.

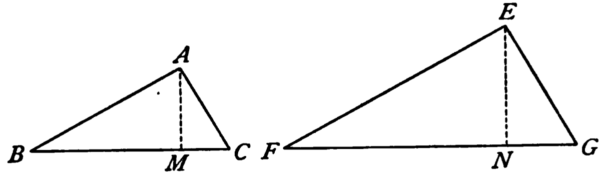



*Suggestion.* Connect $B$ and $C'$. Compare each $\triangle$ with $ABC'$. § 269.

Ex. 213. Prove that the area of a rhombus equals one half the product of its diagonals.

## Proposition VII.

**337. Theorem.** *The ratio of the areas of two similar triangles is equal to the ratio of the squares of their homologous sides, or homologous altitudes.*



**Let A B C and E F G represent two similar triangles, and A M and E N homologous altitudes.**

*To prove* $\frac{\triangle ABC}{\triangle EFG} = \frac{\overline{BC}^2}{\overline{FG}^2}$ *or* $\frac{\overline{AM}^2}{\overline{EN}^2}$.

*Suggestion 1.* The area of $\triangle ABC = \frac{1}{2} BC \times AM$.

2. The area of $\triangle EFG = \frac{1}{2} FG \times EN$.

3. $\therefore \frac{\text{area } \triangle ABC}{\text{area } \triangle EFG} = \frac{\frac{1}{2} BC \times AM}{\frac{1}{2} FG \times EN}$ (§ 202) $=$

$\frac{BC \times AM}{FG \times EN} = \frac{BC}{FG} \times \frac{AM}{EN} = \frac{BC}{FG} \times \frac{BC}{FG}$ $\left(\frac{BC}{FG} = \frac{AM}{EN}.\right.$

Why?$\left.\right) = \frac{\overline{BC}^2}{\overline{FG}^2} = \frac{\overline{AM}^2}{\overline{EN}^2}$.

Give authority for each step.

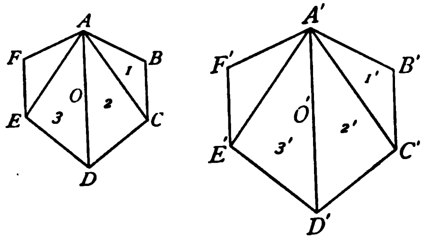
*Therefore —*

---

Ex. 214. Given the area of a trapezoid equal to 104 sq. ft., the altitude equal to 6 ft., and the difference between the bases equal to 2 ft. Find the two bases of the trapezoid.

## Proposition VIII.

**338. Theorem.** *The ratio of the areas of two similar polygons is equal to the ratio of the squares of two homologous sides.*




**Let O and O′ represent two similar polygons.**

*To prove that* $\frac{O}{O'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$, *or the square of the ratio of similitude.*

*Suggestion* 1. Divide the polygons into similar $\triangle$s, 1 similar to 1′, etc. § 309.

2. What does the ratio $\frac{\triangle 1}{\triangle 1'}$ equal in terms of $\frac{AB}{A'B'}$? of $\frac{AC}{A'C'}$? $\frac{\triangle 2}{\triangle 2'}$ in terms of $\frac{AC}{A'C'}$? Compare $\frac{\triangle 1}{\triangle 1'}$ with $\frac{\triangle 2}{\triangle 2'}$. Compare $\frac{\triangle 2}{\triangle 2'}$ with $\frac{\triangle 3}{\triangle 3'}$, etc. § 337.

3. Compare $\frac{\triangle 1}{\triangle 1'}$, $\frac{\triangle 2}{\triangle 2'}$, $\frac{\triangle 3}{\triangle 3'}$, etc.

4. Compare $\frac{O}{O'}$ with any of the ratios in (3) § 284.

5. State the ratio $\frac{O}{O'}$ in terms of homologous sides (Sug. 2).

*Therefore—*

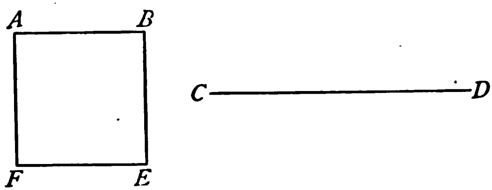
**339.** Corollary.—*The ratio of the areas of two similar polygons equals the ratio of the squares of any two homologous lines.*

Name all the cases in which the ratio of areas has been found.

Notice that ratio of areas is always expressed by the ratio of the products of two factors. If there is a common dimension this factor may be canceled out of both terms of the ratio. State all the cases in which this is true; in which the ratio remains a product. When may this product be stated as a square?

## Proposition IX.

**340. Problem.** *Upon a given line as base, to construct a rectangle equal in area to a given square.*




**Let C D be the given line, and A E the given square.**

*To construct upon C D as a base, a rectangle equal in area to the square A E.*

*Suggestion 1.* Let $X$ be tne required altitude. Hence $AB \times AB = CD \times X$.

*2.* Make a proportion from the above equation, having $X$ for the fourth term. § 263.

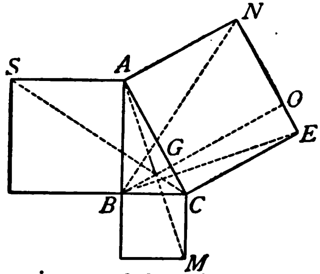
*3.* What problem is involved to find $X$? § 317.

*4.* Construct the rectangle.

*Therefore* —

## Proposition X.

**341. Theorem.** *The square described upon the hypotenuse of a right triangle is equal to the sum of the squares described upon the other two sides.*




**Let A B C represent a right triangle whose hypotenuse is A C, A E the square upon the hypotenuse, and B S and B M the squares on the other two sides.**

*To prove that A E is equal to the sum of B S and B M.*

*Suggestion 1.* Draw *B G* ∥ to *C E*, and extend it to meet *N E* at *O*. Draw *B N*, *B E*, *C S*, and *A M*. What kind of polygons are *C O* and *A O*? Why?

*2.* Compare △s *B C E* and *A C M*.

*3.* Compare the area of △ *A C M* with the area of the square *B M*, and the area of △ *B C E* with the area of the rectangle *C O*. § 332.

*4.* Compare the area of the square *B M* with the area of the rectangle *C O*.

*5.* Compare △s *C A S* and *N A B*.

*6.* Compare the square *B S* with the rectangle *A O*.

*7.* Compare the sum of the areas of the squares *B M* and *B S*, with the sum of the areas of the rectangles *C O* and *A O*; i. e., with the area of the square *C N*.

*Therefore —*

**342.** SCHOLIUM I. This proposition is known as the Pythagorean proposition. It is so named in honor of Pythagoras, who is supposed to have given the first demonstration of it.

**343.** SCHOLIUM II. The "square of a line" is an expression often used instead of the square described upon a line, and when so used means a plane surface in the form of a square whose side is the given line. But when this is used algebraically to express the area of the square surface whose side is the given line, it must be interpreted as in § 329.

(*a*) The Pythagorean proposition is often stated: The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides. It is frequently expressed as an algebraic equation, thus:

$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$, in which the terms may be interpreted as pure numbers.

---

Ex. 215. If the diagonals of a quadrilateral intersect at right angles, show that the area of the quadrilateral is one-half the area of a rectangle whose sides are equal to the diagonals of the quadrilateral.

Ex. 216. To draw a common tangent to two given circles.

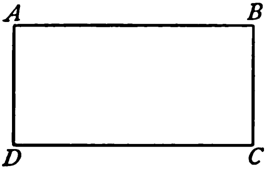
Ex. 217. If three equal circles are tangents to one another, the lines joining their centers form an equilateral triangle.

Ex. 218. Draw a circle with its center at a given point tangent to a given circle. When is there only one solution? When two?

Ex. 219. Construct a triangle with an area equal to 9 times the area of a triangle whose sides are 6, 7, and 9.

## Proposition XI.

**344. Problem.** *To construct a square equal in area to a given rectangle.*




**Let A C be a given rectangle.**

*To construct a square equal in area to rectangle A C.*

*Suggestion 1.* Let $X$ represent the side of the square required.

*2.* Express the area of $AC$ in terms of base and altitude; of the required square in terms of $X$.

*3.* Make an equation, and from it a proportion, and find $X$. §§ 263, 319.

NOTE—There are three previous propositions by which the value of $X$ can be found. It may be time well used by pupils to work out more than one solution.

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Ex. 220. Construct a square equal in area to a given trapezoid.

Ex. 221. Construct a square equal in area to a given triangle.

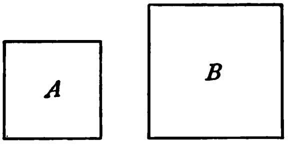
Ex. 222. Construct a square equal in area to a given parallelogram.

Ex. 223. If the hypotenuse of a right triangle is 15 ft. and the ratio of the legs is $\frac{3}{4}$, what is the area?

Ex. 224. Prove that the area of an equilateral triangle constructed on the hypotenuse of a given right triangle, is equal to the sum of the areas of the equilateral triangles constructed on the other two sides of the given right triangle. §§ 341 and 337.

PROPOSITION XII.

**345. Problem.** *To construct a square whose area is equal to the sum of the areas of two given squares.*




**Let A and B be two given squares.**

*To construct a square whose area is equal to the sum of the areas of A and B.*

*Suggestion.* What proposition is suggested by the theorem?

PROPOSITION XIII.

**346. Problem.** *To construct a square whose area is equal to the difference of the areas of two given squares.* § 343 (*a*).

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Ex. 225. Construct a square whose area is equal to the sum of the areas of three given squares.

Ex. 226. Upon a given line as a base, construct a rectangle whose area is equal to the sum of the areas of a given square, a given trapezoid, and a given triangle.

Ex. 227. The area of a square inscribed in a circle is one-half the area of a square circumscribed about the same circle.

Ex. 228. Construct a parallelogram having a given base and a given angle, whose area is equal to a given rectangle.

Ex. 229. Find the dimensions of a rectangle whose perimeter is 96 in. and whose area is 40 sq. in.

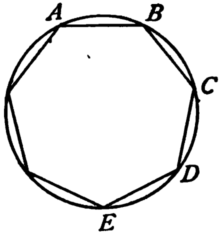
# CHAPTER V.

## REGULAR POLYGONS AND CIRCLES.

347. **A regular polygon** is a polygon which is both equilateral and equiangular. The equilateral triangle and square are regular polygons.

PROPOSITION I.

348. **Theorem.** *An equilateral polygon inscribed in a circle is a regular polygon.*




**Let A D represent an equilateral polygon inscribed in a circle.**

*To prove that A D is a regular polygon.*

*Suggestion 1.* What remains to be proved to make *A D* a regular polygon according to the definition, § 347?

*2.* Compare $\angle$s *A B C*, *B C D*, etc. Give auth.

*3.* Apply the definition of a regular polygon to *A D*.

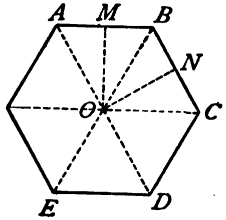
*Therefore* —

349. Corollary. *A regular polygon may have any number of sides.*

If the circumference of a circle be divided into any number of equal parts, the lines joining the points of division form an inscribed equilateral polygon. How does the number of sides of the polygon compare with the number of parts into which the circumference is divided?

## Proposition II.

**350. Theorem.** *A circle can be circumscribed about a regular polygon. A circle can be inscribed in a regular polygon.*




**Let A D represent a regular polygon.**

I. *To prove that a circle can be circumscribed about A D.*

*Suggestion 1.* At $M$ and $N$, the middle points of two adjacent sides, erect ⊥s and extend them until they meet, as at $O$. Why do they meet? $O$ is the center of a circumference which passes through $A$, $B$, and $C$. Why?

2. To prove $O A = O D = O E$, etc.

(*a*) Join $O$ with each vertex of the polygon. Compare $O C$ with $O B$ (Auth.); $\angle O B A$ with $\angle O C D$ (Auth.); $\triangle O B A$ with $\triangle O C D$, and hence, $O A$ with $O D$.

(*b*) The circumference through $A$, $B$, and $C$ also passes through $D$. Why?

(*c*) In a similar manner show that the same circumference passes through $E$, etc. Can a ⊙ be circumscribed about the polygon $A D$?

II. *To prove that a circle can be inscribed in the polygon A D.*

*Suggestion 1.* Compare the distances of the various sides of the polygon from $O$. § 188.

*2.* Complete the demonstration.

*Therefore* —

QUERY: How many sides has polygon $A D$? See the theorem.

351. **The radius of a regular polygon** is the radius of the circumscribed circle.

352. **The apothem of a regular polygon** is the radius of the inscribed circle.

353. **The center of a regular polygon** is the center of the inscribed or circumscribed circle.

354. **The angle at the center of a regular polygon** is the angle formed by two radii drawn to two adjacent vertices of the polygon.

From the definitions just given the following corollaries may be deduced. Prove them.

355. COROLLARY I. *The angle at the center of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

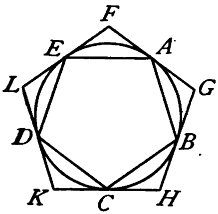
356. COROLLARY II. *An interior angle of a regular polygon is equal to the sum of all the interior angles of the polygon divided by the number of sides of the polygon.*

357. COROLLARY III. *The angle at the center of a regular polygon is the supplement of an interior angle of the polygon.*

358. COROLLARY IV. *The radius of a regular polygon bisects the angle of the polygon to which it is drawn.*

### PROPOSITION III.

**359. Theorem.** *If a circumference is divided into any number of equal parts, the tangents drawn through the points of division form a regular circumscribed polygon.*




**Let the circumference A B C, etc., be divided into equal parts, A B, B C, etc., and at the points A, B, etc., let tangents be drawn forming the polygon G H K, etc.**

*To prove that G H K, etc., is a regular polygon.*

*Suggestion 1.* Compare ∠s $G$, $H$, $K$, etc. Ex. 130.

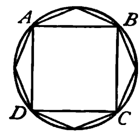
*2.* Compare lines $FA$, $AG$, $GB$, $BH$, etc. Exs. 64 and 125.

*3.* Compare lines $FG$, $GH$, $HK$, etc.

*4.* Apply definition of a regular polygon.

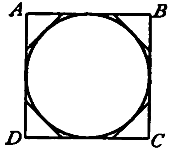
*Therefore —*

360. COROLLARY I. *If the vertices of a regular inscribed polygon are connected with the middle points of the arcs subtended by the sides, a regular inscribed polygon of double the number of sides is formed.* § 179.




361. COROLLARY II. *The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides.*

362. Corollary III. *If a regular polygon is circumscribed about a circle, and if tangents are drawn at the middle points of the intercepted arcs, a regular circumscribed polygon of double the number of sides is formed.* § 359.




363. Corollary IV. *The perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.*

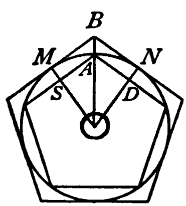
## Proposition IV.

364. **Theorem.** *If a regular polygon is inscribed in a circle, and if tangents are drawn at the middle points of the arcs subtended by the sides of the inscribed polygon:*

I. *A regular circumscribed polygon is formed.*

II. *The sides of the regular circumscribed polygon are parallel to the sides of the inscribed polygon, each to each.*

III. *The vertices of the circumscribed polygon lie in the radii extended of the inscribed polygon.*




**Let S A D represent a regular inscribed polygon; M, N, etc., the middle points of the arcs subtended by the sides; M B, B N, etc., tangents drawn through M, N, etc.**

*To prove* —

I. *M B N is a regular polygon.*

II. *M B is parallel to S A, B N is parallel to A D, etc.*

III. *B lies in O A extended, etc.*

*Suggestion for* I. § 359.

*Suggestion for* II. Draw a radius ⊥ to a side of the inscribed polygon and extend.

*Suggestion for* III. Draw *O M* and *O N* to adjacent points of tangency. Draw *O B* and *O A* to the vertices of the circumscribed and inscribed polygons, respectively, included between those points of tangency. Compare △s *S A O* and *D A O*; △s *M B O* and *N B O*.

*2.* ∴ ∠*M O N* is bisected by *A O* and also by *B O*. Why?

*3.* Complete the demonstration. § 47.

*Therefore*—

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Ex. 230. The ratio of similitude of two similar polygons is $\frac{5}{7}$, and the sum of their areas is 518 sq. in. Find the area of each polygon. § 279.

Ex. 231. Find the base of a rectangle whose area is 108 sq. ft. and whose altitude is 6 ft. Compare your method with that of § 340.

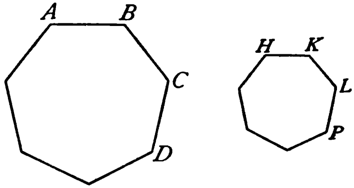
Ex. 232. Find the area of a right triangle whose hypotenuse is 1 ft. 8 in., and one of whose legs is 1 ft.

Ex. 233. The area of a polygon that circumscribes a circle equals one-half the product of its perimeter and the radius of the circle.

Ex. 234. If the middle points of two adjacent sides of a parallelogram be joined, a triangle is formed equal in area to one-eighth of the area of the parallelogram.

## Proposition V.

**365. Theorem.** *Regular polygons of the same number of sides are similar.*




**Let A D and H P represent two regular polygons of the same number of sides.**

*To prove that A D and H P are similar polygons*

*Suggestion 1.* What must be established to make the polygons similar?

*2.* Compare $\angle A$ with $\angle H$, $\angle B$ with $\angle K$, etc.

*3.* Compare $\frac{A\,B}{B\,C}$ with $\frac{H\,K}{K\,L}$, etc.; $\frac{A\,B}{HK}$ with $\frac{B\,C}{K\,L}$, etc.

Extend the series.

*Therefore* —

---

Ex. 235. What is the locus of centers of circles which are tangent to a given line at a given point?

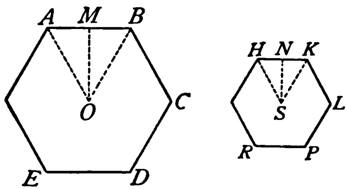
Ex. 236. Two parallel chords of a circle are, respectively, 36 inches and 48 inches long; the radius of the circle is 30 inches. What is the distance between the chords?

Ex. 237. Find the dimensions of a rectangle whose perimeter is 16 in. and whose area is 15 sq. in. Of one whose perimeter is 28 ft. and whose diagonal is 100 ft.

Ex. 238. If a parallelogram be inscribed in or circumscribed about, a circle, the diagonals pass through the center.

## PROPOSITION VI.

**366. Theorem.** *The perimeters of two similar regular polygons have the same ratio as their radii, or their apothems.*




**Let A D and H P represent two similar regular polygons.**

*To prove that* $\frac{A\ B + B\ C + C\ D, \text{etc.}}{H\ K + K\ L + L\ P, \text{etc.}} = \frac{O\ A}{S\ H} = \frac{O\ M}{S\ N}$.

*Suggestion* 1. Compare the ratio of the perimeters with the ratio of any two homologous sides, as with $\frac{A\ B}{H\ K}$. Compare $\frac{A\ B}{H\ K}$ with $\frac{O\ A}{S\ H}$, with $\frac{O\ M}{S\ N}$. Give auth. § 305.

Complete the demonstration.

*Therefore* —

## PROPOSITION VII.

**367. Theorem.** *The areas of two similar regular polygons have the same ratio as the squares of their radii, or the squares of their apothems.*

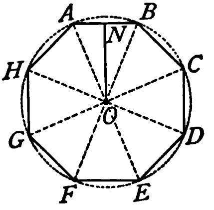
**Let O and S represent two similar regular polygons.**

*To prove that* $\frac{O}{S} = \frac{A\ O^2}{H\ S^2} = \frac{M\ O^2}{N\ S^2}$.

*Suggestion.* The same method may be used as in § 366. See § 338.

## PROPOSITION VIII.

368. **Theorem.** *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*




**Let A D be a regular polygon, O N its apothem, and A B C, etc., its perimeter.**

*To prove that the area of the polygon is equal to one-half the product of A B C, etc., and O N.*

*Suggestion 1.* Circumscribe a circle, and draw radii to the vertices of the polygon.

2. Compare the distances from the center to the sides of the polygon. Give auth.

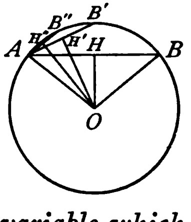
3. What is the area of △ *A O B*? Of △ *B O C*? etc.

4. What is the sum of the areas of all the △s?

*Therefore* —

Work out Sug. 4 algebraically as well as geometrically.

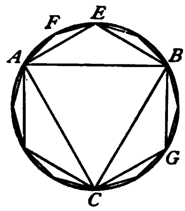
369. A **Lemma** is a proposition, corollary, or postulate inserted out of its natural order for the purpose of immediate use in demonstrating another proposition.




370. **Lemma.** *If the number of sides of a regular inscribed polygon be increased indefinitely the apothem is a variable which approaches the radius as a limit.*

$OH$, $OH'$, $OH''$, etc., is a variable which approaches $OA$, or radius, as its limit.

**371. Lemma.** *If the number of sides of a regular inscribed polygon be increased indefinitely, the perimeter of the polygon is a variable which approaches the circumferance of the circle as a limit.*




Perimeter $ABC$, $AEBG$, etc., $AFE$, etc., is a variable approaching the circumference as its limit.

**372. Lemma.** *If the number of sides of a regular inscribed poloygon be increased indefinitely, the area of the polygon is a variable which approaches the area of the circle as a limit.*

Polygon $ABC$, $AEBG$, etc., $AFE$, etc., is a variable approaching the circle as its limit. § 211 (d) and (e).

Are §§ 371 and 372 true if the words circumscribed be substituted for inscribed?

---

Ex. 239. If one acute angle of a right triangle is 60°, prove that the area of the equilateral triangle constructed on the hypotenuse is equal to the area of a rectangle whose adjacent sides are the two legs of the right triangle.

Ex. 240. If two triangles have two sides of one equal respectively to two sides of another and the included angles supplementary, the triangles are equal in area.

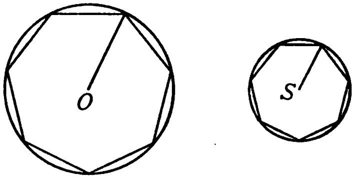
Ex. 241. Draw a circle through a given point tangent to a given line at a given point.

*Suggestion.* Find two loci of the center.

Ex. 242. The diagonals of a parallelogram divide it into four triangles equal in area.

### Proposition IX.

**373. Theorem.** *The circumferences of two circles have the same ratio as their radii.*




**Let O and S represent two circles, R and R′ their radii, and C and C′ their circumferences.**

*To prove that* $\frac{C}{C'} = \frac{R}{R'}$.

*Suggestion 1.* Inscribe in the two circles similar regular polygons. Let $P$ and $P'$ represent their perimeters. Then $\frac{P}{P'} = \frac{R}{R'}$. Give auth.

*2.* From the equation in Sug. 1, $P = \frac{R}{R'} \times P'$. § 270, Note 1.

*3.* Now, let the number of sides of each polygon be indefinitely increased, always, however, keeping the number of sides of one polygon the same as the number of sides of the other polygon. During this change the variable $P$ is always equal to the variable $\frac{R}{R'} \times P'$. Why?

*4.* Since $P$ and $P'$ are variables which approach $C$ and $C'$ respectively, as their limits (§ 371), therefore, $P$ and $\frac{R}{R'} \times P'$, are variables which approach $C$ and $\frac{R}{R'} \times C'$, respectively as their limits. Why? § 214 (*a*).

5. Compare $C$ and $\frac{R}{R'} \times C'$. Give auth.

6. Compare $\frac{C}{C'}$ and $\frac{R}{R'}$.

*Therefore* —

QUERY: In Sug. 1, $\frac{P}{P'} = \frac{R}{R'}$. Is $\frac{R}{R'}$ a variable? Why? Is $\frac{P}{P'}$ a variable?

**374.** COROLLARY I. *The circumferences of two circles have the same ratio as their diameters.*

If $D$ and $D'$ represent their diameters, $\frac{C}{C'} = \frac{D}{D'}$. Auth.

**375.** COROLLARY II. *The ratio of the circumference of a circle to its diameter is constant.*

For, by Cor. I, $\frac{C}{C'} = \frac{D}{D'}$, and by alternation $\frac{C}{D} = \frac{C'}{D'}$

This ratio is represented by the Greek letter $\pi$ (pi). Hence, $\frac{C}{D} = \pi$; or, $\frac{C}{2R} = \pi$.

**376.** COROLLARY III. $C = \pi D$; or, $C = 2\pi R$.

---

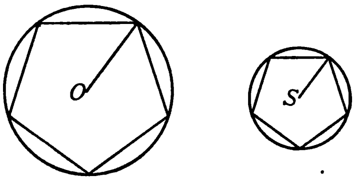
Ex. 243. What is the locus of the vertex of all isosceles triangles upon a given base?

Ex. 244. $A\,B\,C$ is a triangle, and $D$ any point in $B\,C$ extended. Find a point $E$ in $A\,B$ or $A\,B$ extended, such that the area of the triangle $E\,B\,D$ will be equal to the area of the triangle $A\,B\,C$. § 317.

Ex. 245. Draw a line through a given point in a side of a triangle so as to divide the triangle into two parts equal in area.

## PROPOSITION X.

**377. Theorem.** *The areas of two circles have the same ratio as the squares of their radii.*



**Let A and A′ represent the areas, and R and R′ the radii of the circles O and S, respectively.**

*To prove that* $\frac{A}{A'}$ is equal to $\frac{R^2}{R'^2}$.

*Suggestion 1.* Inscribe in the two circles regular polygons of the same number of sides, and let $M$ and $M'$ represent their respective areas.

*2.* Proceed as in § 373.

*Therefore —*

**378.** COROLLARY I. *The areas of circles have the same ratio as the squares of their diameters.*

**379.** COROLLARY II. *The areas of similar sectors have the same ratio as the squares of their radii.*

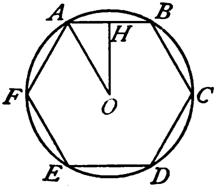
---

Ex. 246. Two tangents to a circle whose radius is 8 inches are drawn from a point 12 inches from the center. Find the length of the chord joining the points of tangency.

Ex. 247. The legs of a trapezoid are each 15 inches, and the bases are 12 inches and 30 inches respectively. Find the area of the trapezoid.

## PROPOSITION XI.

380. **Theorem.** *The area of a circle is equal to one half the product of its circumference and radius.*




**Let O represent a circle, R its radius, C its circumference and A its area.**

*To prove that A is equal to one-half the product of C and R.* § 329.

*Suggestion 1.* Inscribe in the circle a regular polygon, and let $P$ denote its perimeter, $M$ its area and $r$ its apothem.

*2.* What is the area of $M$ in terms of $P$ and $r$? § 368.

*3.* If the number of sides of the polygon be indefinitely increased, $M$ is a variable. Why?

*4.* What is its limit? § 372.

*5.* To find the limit of $\frac{1}{2} P \times r$, the area of $M$:

(*a*) $P$ is a variable. Why? What is its limit? Why?

(*b*) $r$ is a variable. Why? What is its limit? Why?

(*c*) $P \times r$ is a variable whose limit is $C \times R$. § 214 (*c*).

(*d*) $\frac{1}{2} P \times r =$ ? Why? § 214 (*a*).

*6.* Compare the limits of the two equal variables, $M$ and $\frac{1}{2} P \times r$.

∴ the area of $O =$ ? Why?

*Therefore —*

381. COROLLARY I. *The area of a circle is equal to $\pi$ times the square of the radius.*

Substituting the value of $C$, (§ 376), $A = \frac{1}{2} C \times R$, $= \frac{1}{2} \times 2\pi R \times R = \pi R^2$.

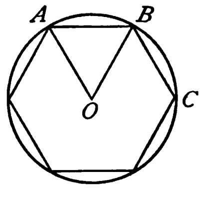
Commit to memory this formula for the area of a circle.

382. Corollary II. *The area of a sector is equal to one-half the product of its radius and arc.*

$\frac{\text{Sec.}}{\text{circle}} = \frac{\text{arc of sec.}}{\text{circumference.}}$ Hence, if a sector is $\frac{1}{n}$ of a circle, its arc is $\frac{1}{n}$ of its circumference; hence its area, $\frac{1}{n}$ of $\frac{1}{2} C \times R = \frac{1}{2}$ arc $\times R$.

## Proposition XII.

383. **Theorem.** *One side of a regular hexagon is equal to the radius of the circumscribed circle.*




**Let A B C, etc., represent a regular hexagon inscribed in a circle whose radius is O A.**

*To prove* $AB = OA$.

*Suggestion 1.* Draw radius $OB$ and prove that $\triangle AOB$ is equiangular and equilateral.

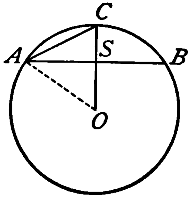
*Therefore* —

---

Ex. 248. Describe a circle through a given point that shall touch a given circle at a given point. Find two loci of the center of the required circle.

## PROPOSITION XIII.

**384. Problem.** *Given the radius of a circle and the side of a regular inscribed polygon, required to find the side of a regular inscribed polygon of double the number of sides.*




**Let O represent a circle, R its radius, A B the side of a regular inscribed polygon and A C the side of a regular inscribed polygon of double the number of sides.**

*To find value of A C in terms of A B and R.*

*Suggestion 1.* Connect $O$ and $A$.

2. In $\triangle ASC$, express $AC$ in terms of $AS$ and $CS$ (§ 341); of $AB$ and $CS$.

3. Express $CS$ in terms of $R$ and $SO$. Express $SO$ in terms of $R$ and $AS$ (§ 341); in terms of $R$ and $AB$.

4. Find value of $CS$ in terms of $R$ and $AB$.

5. Find value of $AC$ in terms of $R$ and $AB$. Reduce to simplest form, viz.: $AC = \sqrt{2R^2 - R\sqrt{4R^2 - \overline{AB}^2}}$

If $R =$ unity, $AC = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$

Partial demonstration:

$$AC = \sqrt{\overline{AS}^2 + \overline{CS}^2} = \sqrt{\tfrac{1}{4}\overline{AB}^2 + \overline{CS}^2} =$$

$$\sqrt{\tfrac{1}{4}\overline{AB}^2 + (R - SO)^2} = \sqrt{\tfrac{1}{4}\overline{AB}^2 + \left(R - \sqrt{R^2 - \tfrac{1}{4}\overline{AB}^2}\right)^2} =$$

$$\sqrt{\tfrac{1}{4}\overline{AB}^2 + R^2 - 2R\sqrt{R^2 - \tfrac{1}{4}\overline{AB}^2} + R^2 - \tfrac{1}{4}\overline{AB}^2} =$$

$$\sqrt{2R^2 - 2R\sqrt{R^2 - \tfrac{1}{4}\overline{AB}^2}} = \sqrt{2R^2 - R\sqrt{4R^2 - \overline{AB}^2}}.$$

## Proposition XIV.

**385. Problem.** *To compute approximately the ratio of the circumference of a circle to its diameter.*

If $A\,C$ (§ 384) is one side of a regular polygon of $n$ sides, $n \times A\,C =$ the perimeter which is an approximation of the circumference of the circumscribed circle, and the greater $n$ is the closer the approximation. Why?

If $A\,C$ is one side of a regular hexagon, $R$ being unity, the perimeter $= 6$. § 383.

If $A\,B$ (§ 384) $=$ one side of a regular hexagon, $A\,C = \sqrt{2 - \sqrt{4 - (1)^2}} = \sqrt{2 - \sqrt{3}} = .51763809$. Hence, the perimeter of a regular twelve-sided polygon $= 12 \times .51763809 = 6.21165708$.

If the value of one side of the regular twelve-sided polygon be substituted for $A\,B$ in the formula, $A\,C$ is the value of one side of a regular twenty-four sided polygon.

By continuing the operation the perimeters of regular polygons of greater and greater number of sides may be found and consequently closer and closer approximations of the circumference in terms of radius.

In this way, the perimeter of a polygon of 768 sides has been computed to be $6.283169 +$.

Dividing this result by the diameter, *i. e.*, by 2 (§ 375), gives $3.141584 +$, as an approximate value of the ratio of the circumference of a circle to its diameter, an approximate value of $\pi$.

The approximate value usually used is 3.1416.

Therefore, $\pi = 3.1416$ approximately. *Q. E. D.*

(*a*) Some of the computed results are given in the following table:

| No. Sides. | One Side. | Perimeter. |
|---|---|---|
| 6 | 1 | 6. |
| 12 | $\sqrt{2-\sqrt{3}} = .51763809$ | 6.21165708 |
| 24 | $\sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238$ | 6.26525722 |
| 48 | $\sqrt{2-\sqrt{4-(.26105238)^2}} = .13080626.$ | 6.27870041 |
| 96 | $\sqrt{2-\sqrt{4-(.13080626)^2}} = .06543817.$ | 6.28206396 |
| 192 | $\sqrt{2-\sqrt{4-(.06543817)^2}} = .03272346$ | 6.28290510 |

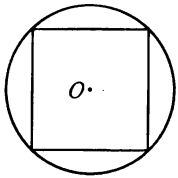
386. COROLLARY. *As an approximate value of $\pi$ has been found, the area of a circle may be found approximately in terms of its radius. The approximate value is found by multiplying the square of the radius by 3.1416.* § 381.

387. SCHOLIUM. Archimedes (born 287 B.C.) found an approximate value of $\pi$. He proved that its value is between $3\frac{1}{7}$ and $3\frac{10}{71}$. The former of these two values is often used as an approximate value of $\pi$ when great accuracy is not required.

In modern times the value of $\pi$ has been computed to a large number of decimal places. Clausen and Dase, independently of each other, computed the value to the two-hundredth decimal place. Other computers have given the value to over five hundred decimal places, but their results have not been verified. The number is incommensurable, and cannot be expressed exactly by any number of decimal places.

### Proposition XV.

**388. Problem.** *To inscribe a square in a given circle*




**Let O be a given circle.**

*To inscribe a square in the circle O.*

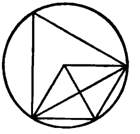
*Suggestion.* In any square, at what ∠ do the diagonals intersect?

389. Corollary. *By bisecting the arcs subtended by the sides of the square, and joining each point of division with the two adjacent vertices, a regular inscribed octagon is formed.*

---

Ex. 249. Measure the diameter and circumference of several articles in the form of a circle, as the end of a barrel, tin cup, etc., and divide the latter by the former to see how an approximation of $\pi$ can be obtained.

Ex. 250. Compute the length of a perimeter of twenty-four sides and verify by the table, P. 211.



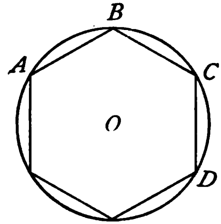
Ex. 251. Show that one side of an equilateral triangle inscribed in a circle is equal to $R\sqrt{3}$, or if $R$ equals unity, $\sqrt{3}$.

Ex. 252. Use the formula, (§ 384), $\sqrt{2-\sqrt{4-ab^2}}$ in the case of a regular inscribed equilateral triangle to prove that the value of one side of a regular hexagon is one.

*Suggestion.* $AB = \sqrt{3}$. (Ex. 251.)

## PROPOSITION XVI.

**390. Problem.** *To inscribe a regular hexagon in a given circle.*




**Let O be a given circle.**

*To inscribe a regular hexagon in the circle O.*

*Suggestion.* § 383.

**391.** COROLLARY I. *By joining the alternate vertices of the regular inscribed hexagon in order, an equilateral triangle is inscribed in the circle.*

**392.** COROLLARY II. *By bisecting the arcs subtended by the sides of a regular inscribed hexagon, and joining the points of division with the adjacent vertices of the hexagon, a regular dodecagon is inscribed in the circle.*

**393.** GENERAL SCHOLIUM. Methods have already been given of inscribing in a circle a regular polygon of 3, 4, 5, 6, 8, or 10 sides (see § 741). Any regular inscribed polygon being given, a regular inscribed polygon of double the number of sides can be formed by bisecting the arcs subtended by the sides and joining the points of division to the adjacent vertices of the given polygon. Hence, by means of the inscribed square regular polygons of 8, 16, 32, etc., sides can be inscribed; by means of the regular inscribed hexagon, regular polygons of 12, 24, 48, etc., sides can be inscribed; by means of the regular inscribed decagon regular polygons of 20, 40, 80, etc., sides can be inscribed. There is still one more set of polygons which can be inscribed in a circle. For, if from any point on the circumference of a circle a chord be drawn equal to one side of a regular inscribed hexagon, and from the same point another chord be drawn equal to the side of a regular inscribed decagon, the first chord subtends an arc which is one-sixth of the circumference, and the second chord an arc which is one-tenth of the circumference, and the difference between these two arcs is one-fifteenth of the circumference. Hence, the chord joining the extremities of the two chords previously drawn is one side of a regular inscribed polygon of fifteen sides, and from this figure regular polygons of 30, 60, etc., sides can be inscribed.

Until the beginning of the present century it was supposed that the polygons already enumerated were the only ones which could be inscribed by elementary geometry, but in a work published in 1801, Gauss proved, by means of the ruler and dividers only, that it is possible to inscribe regular polygons of 17 sides, of 257 sides, and in general, of any number of sides which can be expressed by $2^n+1$, $n$ being an integer, provided that $2^n+1$ is a prime number.

---

Ex. 253. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

Ex. 254. Construct a triangle equal in area to a given triangle, the base, an angle adjacent and the distance from the vertex to the middle point of the base being given.

# SOLID GEOMETRY.

## CHAPTER VI.

### LINES AND PLANES.

**Definitions.**

394. **Geometry of three dimensions** treats of figures whose parts are not confined to a single plane.

(*a*) Geometry of three dimensions is also called **solid geometry** and **geometry of space.**

(*b*) The results of plane geometry furnish the basis for investigations in geometry of three dimensions, but it must be remembered that the statements of **plane** geometry are made with respect to figures which are entirely in one plane, and are not necessarily true in geometry of three dimensions. For instance: It has been proved in plane geometry that only one perpendicular can be erected to a line at a given point. In solid geometry many perpendiculars can be so erected. This may be illustrated by the spokes of a wheel all perpendicular to its axis. Therefore, in geometry of three dimensions, the theorems of plane geometry must not be applied unless the reference is to parts of a figure all of which are in one plane; but those theorems may be applied first to one plane, then to another, and so on.

In reasoning from plane geometry to geometry of three dimensions, the following axiom is used:

**395. Axiom 19.** The relations of the parts of a figure or figures in one plane are not changed by moving the plane containing the figures from one position to another.

NOTE.—In solid geometry even more than in plane the tendency is to fail to give authority for more or less of the statements in the demonstrations, owing to their greater length. Great care should be exercised to see that authority is fully and freely given, for therein lies in great part the rigor of the demonstration. (See Suggestion to Teachers.)

**396.** A plane is indefinite in extent. From the definition (§ 10) it follows that if a straight line of a plane is extended indefinitely it must always lie in the plane.

**397.** A plane is **determined** when it is exactly located, or is distinguished from every other plane. A plane is determined by lines or points when it is the only plane which contains those lines or points.

**398.** A plane **embraces** a line when the line lies wholly in the plane. The plane is then said to be **passed through the line.**

**399.** The **intersection of two planes** is that portion of them which is common to both.

---

Ex. 255. Draw a line through the vertex of a triangle so as to divide the triangle into parts whose areas have the ratio 2 to 5. § 333.

Ex. 256. Two circles are tangent externally; locate a line of a given length so that it shall pass through the point of contact and have its extremities in the circumferences of the circles.

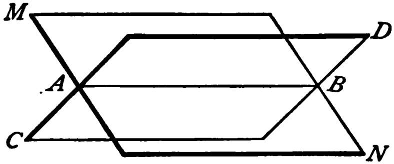
Ex. 257. Given one leg of a right triangle and the radius of the inscribed circle, construct the triangle.

Ex. 258. Inscribe a circle in a given rhombus.

Ex. 259. One side of an inscribed square $= R\sqrt{2}$.

## PROPOSITION I.

**400. Theorem.** *Two planes can embrace the same straight line.*




**Let M N and C D represent two planes.**

*To prove that M N and C D can both embrace the same straight line.*

*Suggestion 1.* Let *M N* and *C D* intersect and let *A* and *B* be two points in their intersection. Connect *A* and *B* by a straight line.

*2.* Where does line *A B* lie in respect to each plane? § 10.

*Therefore* —

**401.** COROLLARY I. *A plane can be made to occupy different positions and still embrace the same line.*

**402.** COROLLARY II. *A straight line does not determine a plane.* § 397.

**403.** When a plane takes in succession the various possible positions while a given line of the plane embraces a given line, as *A B*, the plane is said to **revolve about A B.**

---

Ex. 260. The apothem of an inscribed square $= \frac{R}{2}\sqrt{2}$.

Ex. 261. The altitude of an equilateral triangle is equal to one and one-half times the radius of the circumscribed circle.

## Proposition II.

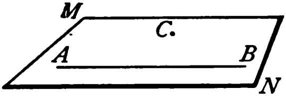
**404. Theorem.** *A plane is determined —*

I. *By a straight line and a point without the line.*

II. *By three points not in a straight line.*

III. *By two intersecting straight lines.*

IV. *By two parallel straight lines.*




**I. Let A B represent a straight line, and C a point not in the line A B.**

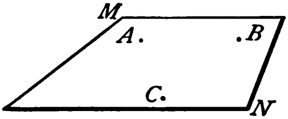
*To prove that the line A B and the point C determine a plane.*

*Suggestion 1.* Through the line *A B* pass a plane *M N* and let it revolve about *A B* as an axis until it contains the point *C*. § 403.

2. How much can the plane be revolved either way about *A B*, and still contain point *C*? Why? §§ 3 and 4.

3. How many planes can embrace the given line and point? Do *A B* and *C* determine the plane *M N*? § 397.

*Therefore —*

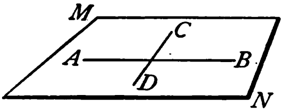



**II. Let A, B and C represent three points not in one straight line.**

*To prove that the points A, B and C determine a plane.*

*Suggestion.* Connect two of the points.

*Therefore —*




**III. Let A B and C D represent two intersecting lines.**

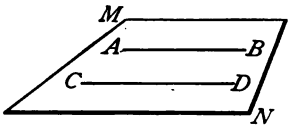
*To prove that A B and C D determine a plane.*

*Suggestion 1.* Pass a plane, *M N*, through the line *A B* and some point of *C D* not in *A B*.

2. How many planes can occupy this position? Why?

3. Where is *C D* with respect to the plane? Why? § 10.

*Therefore —*




**IV. Let A B and C D represent two parallel lines.**

*To prove that A B and C D determine a plane.*

*Suggestion 1.* In how many planes do two ∥ lines lie? § 101.

2. How many different planes may be passed through one of the lines and one point of the other line?

3. Where does the plane of the ∥ lines lie with respect to this plane?

*Therefore—*

---

Ex. 262. From a point without a circle two tangents are drawn which, with the chord connecting the points of contact, form an equilateral triangle whose side is 18 inches. Find the diameter of the circle.

Ex. 263. Two posts set 24 feet apart on level ground are 8 and 12 feet high, respectively. How far apart are their tops?

**405.** A line is **perpendicular to a plane** when it is perpendicular to any line in the plane drawn through its foot.

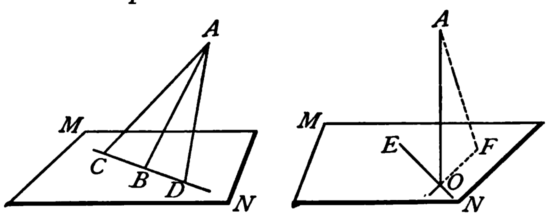
(*a*) The plane is then perpendicular to the line.

**406.** COROLLARY. *When a line is perpendicular to a plane it is perpendicular to every line in the plane drawn through its foot.*

**407.** A line is **oblique to a plane** when it is oblique to one or more lines in the plane drawn through its foot.

## PROPOSITION III.

**408. Theorem.** *From a given point without a plane one, and only one, perpendicular can be dropped to the plane, and the perpendicular is the shortest line from the point to the plane.*




**Let M N represent the given plane, and A the given point without the plane.**

*To prove that from A, one and only one perpendicular can be drawn to the plane M N, and that the perpendicular is shorter than any other line from A to the plane M N.*

*Suggestion 1.* (*a*) Of all lines from *A* to the plane *M N*, either there is one shortest line or a group of equal shortest lines. Suppose *A C* and *A D* are two of a group of equal shortest lines. *A C* and *A D* form a plane. § 404, III.

Connect *C* and *D*.

(*b*) If $A\,C = A\,D$, what kind of $\triangle$ is $A\,C\,D$?

(*c*) From $A$, draw $A\,B \perp$ to $C\,D$. (§ 94.) Compare $A\,B$ and $A\,C$ in respect to length.

(*d*) Then, how many shortest lines are there from $A$ to the plane $M\,N$?

2. (*a*) Let $A\,O$ represent the shortest line from $A$ to the plane $M\,N$, and draw $E\,O$, any line in the plane $M\,N$, through the point $O$.

(*b*) Since $A\,O$ is the shortest line from $A$ to plane $M\,N$, what relation does $A\,O$ sustain to $E\,O$? § 130.

(*c*) Since $E\,O$ is any line drawn through the foot of $A\,O$ in plane $M\,N$, what relation must $A\,O$ sustain to the plane $M\,N$? § 405.

(*d*) Suppose another line from $A$ to plane $M\,N$ can be $\perp$ to $M\,N$, as $A\,F$. Connect $O$ and $F$. Can $A\,F$ be $\perp$ to plane $M\,N$? Why? § 94.

(*e*) How many $\perp$s can be drawn from $A$ to the plane $M\,N$? Why?

*Therefore* —

---

Ex. 264. The apothem of a regular inscribed hexagon is equal to $\frac{1}{2}R\sqrt{3}$, $R$ representing the radius of the circle.

Ex. 265. In a given circle, inscribe an isosceles triangle in which each base angle is one-half the vertical angle.

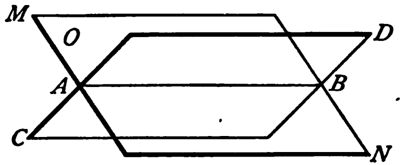
Ex. 266. Divide a circle into segments such that an angle inscribed in one segment shall be three times an angle inscribed in the other segment.

Ex. 267. Find the locus of the center of a circle which passes through two fixed points.

Ex. 268. Find the locus of the points of intersection of tangents to a fixed circle which intersect at a given angle.

PROPOSITION IV.

**409. Theorem.** *The intersection of two planes is a straight line.*




**Let M N and C D represent two planes that intersect and let the straight line A B lie in the intersection.**

*To prove that A B is the intersection of M N and C D.*

*Suggestion 1.* What is the intersection of two planes?
§ 399.

2. Take any point, *O*, of the plane *M N*, not in *A B*. Does this point lie in plane *C D*? Why?

3. ∴ *A B* bears what relation to the intersection?

*Therefore* —

QUERY: What is the difference in meaning in the two statements as used above: *A B* lies in the intersection, and *A B* is the intersection?

**410.** The **locus of a point in space** is a line or lines, a surface or surfaces to which the point is limited and in any point of which it may be found. Compare with the definition of locus of a point in a plane, § 136.

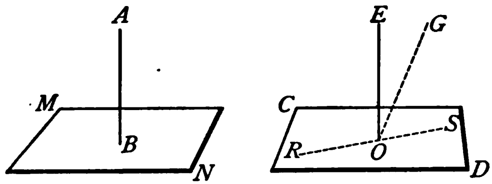
**411.** COROLLARY. *A straight line found in each of two planes is the intersection of the two planes.*

QUERY: What is the locus of a point common to two planes? Apply the definition.

**412.** The point in which a straight line intersects a plane is the **foot of the line.**

### Proposition V.

**413. Theorem.** *At a given point in a plane, one perpendicular, and only one, can be erected to the plane.*




**Let C D represent a plane, and O a given point in the plane.**

*To prove that one, and only one, perpendicular can be erected to the plane C D at the point O.*

*Suggestion 1.* Let $M N$ represent another plane, and $A B$ a $\perp$ from $A$ to the plane $M N$. § 408.

2. Place the plane $M N$ in the plane $C D$, so that $B$ is upon $O$; then revolve plane $M N$ until two more points of $M N$ lie in the plane $C D$. (§ 82, Sug. I. § 404, II.) What relation does $A B$ sustain to the plane $C D$? § 395.

*Therefore —*

*Suggestion.* If more than one $\perp$ can be erected, let $O E$ and $O G$ represent two $\perp$s to $C D$ at $O$; $E O$ and $O G$ determine a plane. Let $R S$ be the intersection of the plane $E O G$ with the plane $CD$. What relation do $OE$ and $OG$ sustain to $R S$? Why? Can a second $\perp$ be erected to a plane at a given point in the plane? Give auth. § 46.

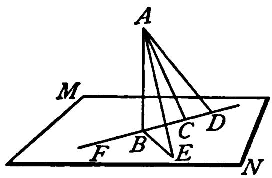
*Therefore —*

Query: Why should $R S$ be taken as the intersection of the planes? 394 (*b*).

Note.—The section references are not expected to be used except in verification of the student's authority, or when he is unable to find an authority for himself. The student would do well to take pride in seeing how few references he needs to use.

## PROPOSITION VI.

**414. Theorem.** *If, from any point in a perpendicular to a plane, oblique lines be drawn, those which meet the plane at equal distances from the foot of the perpendicular are equal; and, of two unequal oblique lines, that which meets the plane at the greater distance from the foot of the perpendicular is the greater.*




**Let M N represent a plane, A B a perpendicular to the plane, A D, A C and A E oblique lines meeting the plane M N at the points D, C and E, respectively, so that B C is equal to B E, and B D is greater than B E.**

I. *To prove that oblique lines A C and A E are equal.*

I. *Suggestion 1.* What relation does *A B* sustain to *B C*, and also to *B E*? Why? § 406.

*2.* Compare $\triangle$s *A B C* and *A B E*; *A C* and *A E*.

II. *To prove that A D is longer than A E.*

I. *Suggestion.* Compare *A C* and *A D* (§ 129, III.); *A C* and *A E*; *A E* and *A D*.

*Therefore* —

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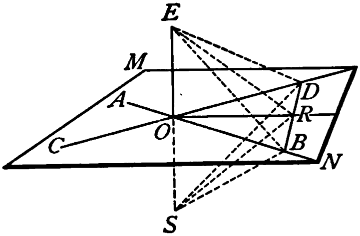
Ex. 269. From any point in the perpendicular to a plane, equal oblique lines drawn to the plane cut off equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater cuts off the greater distance from the foot of the perpendicular.

## PROPOSITION VII.

**415. Theorem.** *If a straight line is perpendicular to two lines of a plane at their point of intersection, it is perpendicular to the plane.*




**Let M N represent a plane, A B and C D two lines of the plane, and E O a perpendicular to A B and C D at their point of intersection O.**

*To prove that O E is perpendicular to the plane M N.*

*Suggestion 1.* Extend *O E* to *S*, making *O S* equal *O E*, and let *O R* represent any line of the plane through the point *O*. Draw the line *B D*, intersecting *O R* at some point, as at *R*. Connect both *E* and *S* with the points *B*, *R* and *D*.

2. In the figure *B E S* compare *B E* and *B S*. In the figure *D E S* compare *D E* and *D S*. Auth. § 129, II.

3. Compare △s *E B D* and *S B D*; ∠s *E B D* and *S B D*. Auth.

4. Compare △s *E B R* and *S B R*; lines *E R* and *S R*. Auth.

5. Compare △s *E O R* and *S O R*; ∠s *E O R* and *S O R*. What relation does *E O* sustain to *O R*? Or relate *E O* and *O R* by Ex. 29 (3).

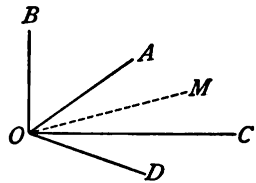
*6*. Since *O R* is any line of the plane through point *O*, what relation does *E O* sustain to the plane? § 405.

*Therefore* —

QUERIES: How many planes are represented in the figure? Read them. Why prove *E B* equal to *S B*? Why prove angles *E B R* and *S B R* equal?

### PROPOSITION VIII.

**416. Theorem.** (*I.*) *All the perpendiculars to a given line at the same point lie in the same plane, and* (*II.*) *the given line is perpendicular to the plane of the perpendicular lines.*




**Let B O be a given line, and O A, O C, O D, etc., lines perpendicular to B O at O.**

*To prove* (I) *that O A, O C, O D, etc., are in the same plane, and* (II) *that B O is* ⊥ *to the plane of these lines.*

I. *Suggestion 1.* Two of the lines, as *O D* and *O C*, determine the plane *D O C*. Auth. What relation does *B O* sustain to plane *D O C*? § 415.

*2*. *O B* and another of the lines, as *O A*, determine another plane. Auth. Let these planes intersect in *O M*. Why must the planes intersect?

*3*. What relation does *B O* sustain to *O M*? (§ 406.) What relation does *B O* sustain to *O A*? Auth. ∴ What relation must *O A* sustain to *O M*? § 46.

*4.* Where is $O\,A$ with respect to plane $D\,O\,C$?

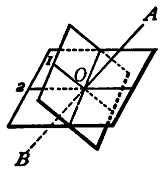
II. *Suggestion.* What relation does $B\,O$ sustain to the plane of $A\,O\,D\,C$?

*Therefore—*

NOTE.—Note carefully the construction of $O\,M$. Would the following answer the purpose? Let $O\,M$ be any line through $O$ in plane $D\,O\,C$? Why? § 394 (*b*).

**417.** COROLLARY I. *Through a given point in a straight line only one plane can be drawn perpendicular to the line.*

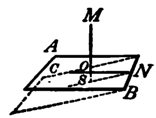
Suppose two planes can both be ⊥ to $A\,B$ at a point, as at $O$. Pass a plane through $A\,B$ cutting both planes elsewhere than their intersection, as in *1* $O$ and *2* $O$. What relation must $A\,B$ bear to *1* $O$ and *2* $O$?




§§ 406, 394 (*b*).

**418.** COROLLARY II. *From a given point without a line, only one plane can be drawn perpendicular to the given line.*

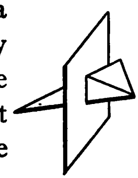
If two planes can be drawn through a point, as $N$, ⊥ to a line, as $M\,S$, let them be represented by $A\,B$ and $C\,B$.




Connect $N$ with the feet of the ⊥ to the planes. What relation do $O\,N$ and $S\,N$ sustain to $M\,S$? Why? What do you think of the possibility of drawing two planes from the same point ⊥ to a line? Give auth.

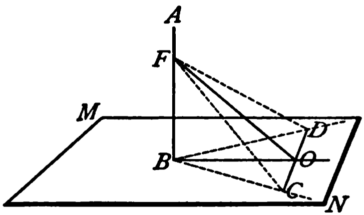
---

Ex. 270. If a plane is perpendicular to a straight line at its middle point. (1) Every point in the plane is equally distant from the extremities of the line. (2) Every point out of the plane is unequally distant from the extremities of the straight line.



## PROPOSITION IX.

**419. Theorem.** *If, from the foot of a perpendicular to a plane, a line is drawn perpendicular to any line of the plane, and, from this point of intersection, a line is drawn to any point of the perpendicular to the plane, the last line is perpendicular to the line of the plane.*




**Let M N represent a plane and A B a perpendicular to the plane. Let C D represent any line of the plane M N. Let B O be perpendicular to D C at O, and let O be joined to F, any point in A B.**

*To prove that F O is perpendicular to C D.*

*Suggestion 1.* On the line *D C*, take *O D* equal to *O C*. Connect *F* with *C* and *D*, *B* with *C* and *D*.

*2.* In △ *B C D*, compare *B C* and *B D*. Auth.

*3.* Compare *F C* and *F D*. § 414 (I.).

*4.* What relation does *F O* sustain to *D C*? § 89 or Ex. 29 (3).

*Therefore* —

Make a physical representation of this theorem using wire or other material. Show that the theorem is always true as *F* moves along *A B*.

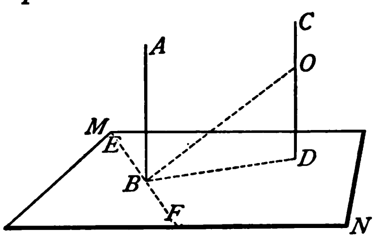
**420. A straight line is parallel to a plane** when the line and plane cannot meet however far they may be extended.

(*a*) The plane is then parallel to the line.

**421. Two planes are parallel** when they cannot meet however far they may be extended.

## PROPOSITION X.

**422. Theorem.** *If one of two parallel straight lines is perpendicular to a plane, the other is also perpendicular to the plane.*




**Let A B and C D be two parallel lines, meeting the plane M N in the points B and D, respectively, and let C D be perpendicular to the plane M N.**

*To prove that A B is perpendicular to the plane M N.*

*Suggestion 1. A B* and *C D* determine a plane which intersects *M N* in *B D*. Why? Connect *B* with any point of *C D*, as *O*. Draw *E F* in the plane *M N* ⊥ to *B D*.

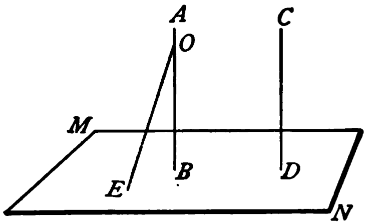
*2.* What relation does *E F* sustain to *B O* (§ 419)? To *B D*? To the plane *A D*? § 415.

*3.* What relation does *E F* sustain to *A B*? Or *A B* sustain to *E F* (§ 406)? *A B* to *B D* (§ 109)? *A B* to plane *M N*? Auth.

*Therefore* —

## Proposition XI.

423. **Theorem.** *If two lines are perpendicular to the same plane, they are parallel.*




**Let the two straight lines A B and C D be perpendicular to the plane M N.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* If *A B* is not ∥ to *C D*, draw, through any point, *O*, of *A B* a line, *O E*, ∥ to *C D*.

*2.* What relation does *O E* sustain to *M N*? § 422.

*3.* § 408.

*Therefore* —

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Ex. 271. What is the locus of a point at equal distances from two given points. Ex. 270.

Ex. 272. Prove the converse of Proposition IX, § 419, in which *F O* is given perpendicular to *C D*. Required to prove *B O* is ⊥ to *C D*.

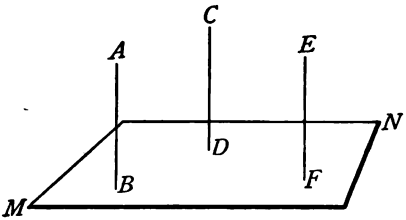
*Suggestion.* Draw *B M* ⊥ to *C D* and connect *M* and *F*.

Ex. 273. What is the locus of the foot of an oblique line of constant length drawn from a point in a perpendicular to a plane?

Ex. 274. What is the area of a sector whose arc is $\frac{1}{6}$ of the circumference, in a circle whose radius is 15 in. § 382.

## PROPOSITION XII.

**424. Theorem.** *If two straight lines are parallel to a given straight line, they are parallel to each other.*




**Let A B and C D each be parallel to E F.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* Draw a plane, $M N$, $\perp$ to $E F$.

2. What relation does $A B$ sustain to the plane $M N$? $C D$ to plane $M N$? $A B$ to $C D$?

*Therefore* —

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Ex. 275. Construct three equal circles which shall be tangent to one another and to a given circle. Consider two cases: (1) that in which the three circles are within the given circle; and (2) that in which the three circles are without the given circle.

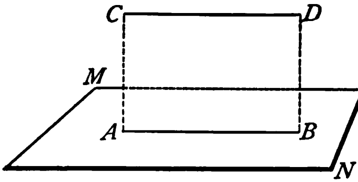
Ex. 276. The hypotenuses of three isosceles right triangles form a right triangle. Prove that one of the isosceles triangles is equal to the sum of the other two.

Ex. 277. Find the locus of the center of a circle tangent to two intersecting straight lines.

Ex. 278. If any two polygons whatever be circumscribed about the same circle prove that their areas have the same ratio as their perimeters.

## PROPOSITION XIII.

**425. Theorem.** *If a line is parallel to a line of a plane, it is parallel to the plane.*




**Let M N embrace A B, and let C D be parallel to A B.**

*To prove that the line C D and the plane M N are parallel.* § 420.

*Suggestion 1. A B* and *C D* determine a plane *C B*. Why?

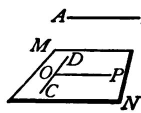
*2.* What is the intersection of the planes *M N* and *C B*? Why? § 406.

*3.* If *C D* meets the plane *M N*, it must meet it in the line *A B*. Why? § 396.

*4.* Is it possible for *C D* to meet the plane *M N*? Why? § 101.

*Therefore —*

**426.** COROLLARY. *Through a given straight line, a plane can be passed parallel to any straight line which does not intersect the given line.*



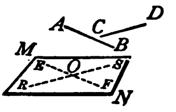

*To prove that a plane can be passed through C D parallel to A B, if A B does not intersect C D.*

*Suggestion.* From any point in *C D*, draw a line ∥ to *A B*, as the line *O P*. The plane determined by *C D* and *O P* bears what relation to *A B*? Why?

QUERY: How many such planes can be passed?

2. Suppose *A B* and *C D* are in the same plane, answer query?

427. COROLLARY II. *A plane can be passed parallel to any two straight lines through any given point without the lines.*



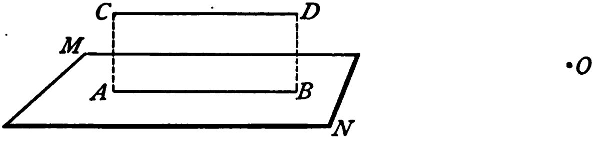

**Let AB and CD be any two lines, and O any point without the lines.**

*Suggestion.* Through *O*, draw two lines ‖ to *A B* and *C D*, respectively.

QUERY: Same as in Cor. I.

## PROPOSITION XIV.

428. **Theorem.** CONVERSE OF PROPOSITION XIII. *If a line is parallel to a plane, any plane that embraces the line and intersects the plane, intersects it in a line parallel to the given line.*




**Let the line C D be parallel to the plane M N and the plane C B, embracing the line C D, intersect M N in the line A B.**

*To prove that A B and C D are parallel.*

*Suggestion 1.* If *C D* and *A B* are not parallel they will meet at some point, as *O*. Why?

2. Can they meet? Why? § 420.

*Therefore—*

**429.** Corollary. *The plane C B may revolve, under the conditions of the theorem, about C D.*

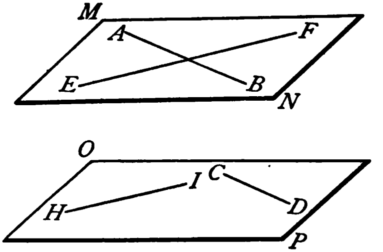
Query: Is there any position in the revolution in which the planes do not intersect?

Note.—The statement if two lines are not parallel they will meet is true in plane geometry, but not necessarily true in solid geometry. Why?

Why is the statement true in Sug. 1?

## Proposition XV.

**430. Theorem.** *If two lines in one plane are respectively parallel to two lines in another plane the planes are parallel.*




**Let A B be parallel to C D and E F to H I.**

*To prove M N, the plane of A B and E F, is parallel to O P, the plane of C D and H I.*

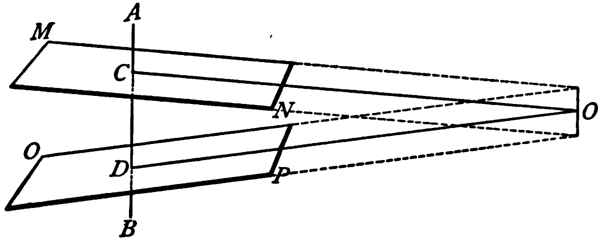
*Suggestion 1.* Plane *O P* bears what relation to *A B*? To *E F*?

*2.* If plane *M N* intersects plane *O P* it must do so only in a line ∥ to *A B* (§ 428) and also only in a line ∥ to *E F*. Why? Why can this not be?

*Therefore* —

## PROPOSITION XVI.

**431. Theorem.** *Planes perpendicular to the same straight line are parallel.*




**Let the planes N M and O P be perpendicular to the straight line A B.**

*To prove that the planes M N and O P are parallel.*

*Suggestion 1.* Let *C* and *D* be the points of intersection of the line *A B* with the planes *M N* and *O P* respectively.

*2.* If the planes *M N* and *O P* intersect, connect any point of their intersection, as *O*, with the points *C* and *D*.

*3.* What relation do these lines sustain to *A B*? Why?

*4.* What is the conclusion?

Or see § 418.

*Therefore* —

---

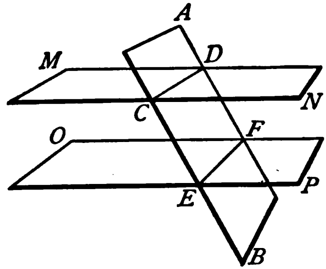
Ex. 279. A straight line and a plane, both perpendicular to the same straight line, are parallel.

Ex. 280. Find the line on which a paper triangle must be folded in order that the vertex may fall upon a given point of the base.

Ex. 281. If a straight line and a plane are parallel, any line parallel to the line is parallel to the plane also.

## PROPOSITION XVII.

432. **Theorem.** *The intersections of two parallel planes with a third plane are parallel lines.*




**Let M N and O P represent two parallel planes, and A B a plane intersecting them in the lines C D and E F.**

*To prove that C D and E F are parallel lines.*

*Suggestion. C D* and *E F* are in the same plane. Why? Can they meet? § 421.

*Therefore* —

See note § 429.

433. COROLLARY. *Parallel lines intercepted between parallel planes are equal.*

---

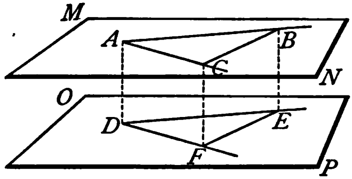
Ex. 282. The locus of points in space equally distant from all points in the circumference of a circle, is a straight line through the center of the circle perpendicular to its plane.

Ex. 283. If two parallel planes intersect two parallel planes, the four lines of intersection are parallel.

Ex. 284. Prove that through a given point in space one and only one line can be drawn parallel to a given line, the given point being without the given line.

## PROPOSITION XVIII.

**434. Theorem.** *If two angles not in the same plane have their sides parallel and lying in the same direction from their vertices they are equal.*




**Let B A C and E D F represent two angles in the planes M N and O P, respectively, having their sides parallel and lying in the same direction from their vertices.**

*To prove that angles B A C and E D F are equal.*

*Suggestion 1.* Take *A C* equal to *D F*; *A B* equal to *D E*; connect *A* and *D*, *C* and *F*, *B* and *E*, *C* and *B*, *F* and *E*.

2. Compare *A D* and *C F* (§ 151); *A D* and *B E*; *C F* and *B E*. Auth.

3. Compare *C B* and *F E*; △s *C A B* and *F D E*; ∠s *A* and *D*. Auth.

*Therefore —*

**435. The distance between two parallel planes** is measured on a line intercepted between them and perpendicular to one of them.

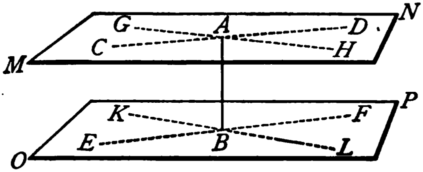
---

Ex. 285. Prove that two parallel planes are everywhere equally distant. Prove the converse of this exercise.

Ex. 286. What is the locus of a point equidistant from two given points, and at the same time equidistant from two other given points?

## Proposition XIX.

**436. Theorem.** *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*




**Let M N and O P represent two parallel planes, and let the line A B be perpendicular to the plane M N.**

*To prove that the line A B is perpendicular to the plane O P.*

*Suggestion 1.* Draw two lines in plane *M N* intersecting at *A*, as *G H* and *C D*. Pass planes through *G H* and *A B*, *C D* and *A B*. Let them intersect plane *P O* in *K L* and *E F*, respectively.

2. What relation does *E F* sustain to *C D*? *K L* to *H G*? Why?

3. What relation does *A B* sustain to *C D*? To *G H*?

4. What relation does *A B* sustain to *E F*? To *K L*? To plane *O P*? § 415.

*Therefore* —

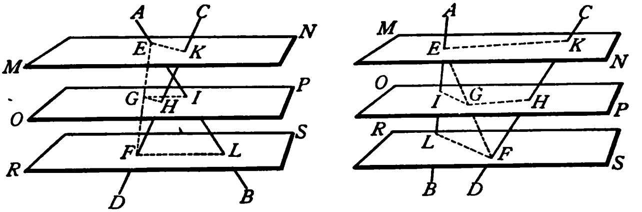
---

Ex. 287. What is the locus of the foot of an oblique line 10 inches long drawn from a point 8 inches from a plane? Compute the area of the figure bounded by the locus.

Ex. 288. Two planes parallel to the same plane are parallel to each other. §§ 436 and 431.

## PROPOSITION XX.

**437. Theorem.** *If three parallel planes are intersected by two lines, the segments of the lines are proportional.*




**Let M N, O P and R S represent three parallel planes, and A B and C D two lines intersecting them, intercepting segments E I and I L on A B, and K H and H F on C D.**

*To prove that* $\frac{E\,I}{I\,L} = \frac{K\,H}{H\,F}$.

*Suggestion 1.* Connect $E$ and $F$, and let the line $E\,F$ intersect the plane $O\,P$ at $G$.

2. The plane $E\,F\,K$ intersects the planes $M\,N$ and $O\,P$ in what lines? The plane $E\,L\,F$ intersects $O\,P$ and $R\,S$ in what lines?

3. What relation between $I\,G$ and $L\,F$? Between $E\,K$ and $G\,H$? Why?

4. Compare the ratios $\frac{E\,I}{I\,L}$, $\frac{E\,G}{G\,F}$ and $\frac{K\,H}{H\,F}$. § 290.

Complete the demonstration.

*Therefore —*

Reconcile the two cuts.

After studying §§ 438–440, illustrate by use of the figure, § 437.

## Dihedral Angles.

438. Two planes which meet or intersect form a **dihedral angle.** The planes *A B* and *C D*, in Fig. 1, meet in the line *A C* and form a dihedral angle.

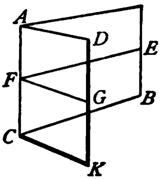



FIG. 1.

439. The intersecting planes, as *A B* and *C D* are the **faces** of the angle, and the line of intersection, as *A C* is its **edge.**

440. A dihedral angle is read by first reading one face, then the edge, then the other face. This can be done by reading four letters, if none are repeated; as *B A C D* or *B C A D*. When but one dihedral angle is formed at an edge, it may be read by reading the edge, as dihedral angle *A C*.

441. **The plane angle of a dihedral angle** is an angle formed by drawing two lines, one in each face, perpendicular to the edge at the same point.

In Fig. 1, if *F E* and *F G* are perpendicular to the edge *A C*, the angle *E F G* is the plane angle of the dihedral angle.

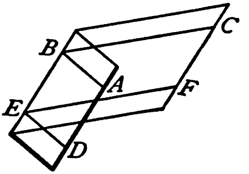
442. Two **dihedral angles are equal** when they can be made to coincide.

443. The magnitude of a dihedral angle does not depend upon the extent of its faces. If a plane be made to revolve from the position of one face about the edge as an axis to the position of the other face, it revolves or turns through the dihedral angle, and the greater the amount of turning the greater the angle.

Compare with definition of plane angle, § 17.

## Proposition XXI.

**444. Theorem.** *All of the plane angles of the same dihedral angle are equal.*

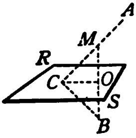



**Let A B C and D E F be two of the plane angles of the same dihedral angle.**

*To prove angle A B C is equal to angle D E F.*

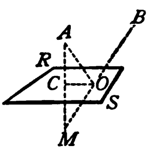
*Suggestion.* § 434.

Ex. 289. Determine a point in a plane, the difference of whose distances from two given points on opposite sides of the plane is the maximum.




*Suggestion.* Drop a ⊥ to the plane from one of the points, as *B*, and extend it to *M*, an equal distance on the other side of the plane. Connect *A* and *M*, and extend the line to meet the plane at *C*. Prove that *C* is the required point.

Ex. 290. Determine a point in a plane, the sum of whose distances from two given points on the same side of the plane is the minimum.

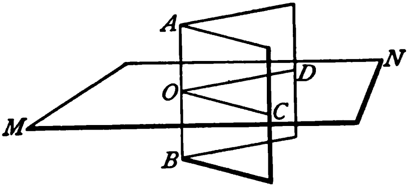



*Suggestion.* Drop a ⊥ to the plane from one of the points *A*, and extend to *M*, an equal distance beyond the plane. Connect *M*, the extremity of the ⊥, with the other point B.

Prove that *O*, the point in which *M B* intersects the plane, is the required point.

### PROPOSITION XXII.

**445. Theorem.** *If a plane be passed perpendicular to the edge of a dihedral angle, its intersections with the faces form the plane angle of the dihedral angle.*




**Let M N be a plane perpendicular to the edge, A B, of a dihedral angle, and let the lines of intersection form the angle C O D.**

*To prove that angle C O D is the plane angle of the dihedral angle.*

*Suggestion 1.* What must be proved to know that $\angle C O D$ is the plane angle of the dihedral?

2. What relation does $A B$ sustain to $O D$ and $O C$? *Therefore* —

QUERY: By what authorities may the plane angle of a dihedral be known?

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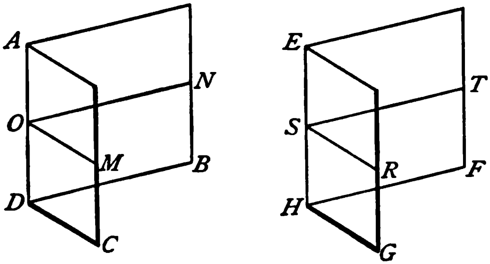
Ex. 291. Find the locus of a point equidistant from two given parallel planes.

Ex. 292. What is the locus of a point equidistant from two given parallel planes, and at the same time equidistant from two other parallel planes?

Ex. 293. If a line and a plane are parallel, a line drawn from any point in the plane parallel to the given line will lie wholly in the plane. § 428.

## Proposition XXIII.

**446. Theorem.** *Two dihedral angles are equal if their plane angles are equal.*




**Let A D and E H be two dihedral angles whose plane angles O and S are equal.**

*To prove that the dihedral angles A D and E H are equal.*

*Suggestion 1.* Place dihedral ∠ *A D* upon dihedral *E H* so that edge *A D* coincides with edge *E H*, point *O* upon *S*. Where do lines *O N* and *O M* lie with respect to the plane *S T R*? Why? § 416.

*2.* Revolve dihedral ∠ *A D* upon *E H* until line *O N* coincides with *S T*. Where does *O M* lie? Why?

*3.* Where do the planes *A B* and *A C* lie with respect to the planes *E F* and *E G*? Why? § 404, III.

*4.* Compare the dihedral ∠s. Give auth.

*Therefore —*

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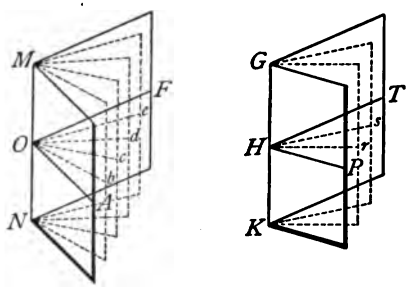
Ex. 294. *A* and *B* are two points equally distant from a plane. Prove that line *A B* is parallel to the plane.

Ex. 295. What is the locus of a point at a given distance from a given plane?

## Proposition XXIV.

**447. Theorem.** *The ratio of two dihedral angles is equal to the ratio of their plane angles.*




**Let M N and G K be two dihedral angles, and A O F and P H T their plane angles.**

*To prove that* $\dfrac{\text{dihedral } MN}{\text{dihedral } GK} = \dfrac{\angle AOF}{\angle PHT}$.

Case I. *When the plane angles are commensurable.*

*Suggestion 1.* Divide the plane ∠s by a common unit of measure. Pass a plane through the edge $MN$ and each of the lines of division of the plane angle $AOF$; through $GK$ and each of the lines of division in the angle $PHT$.

*2.* Let the unit of measure be contained $m$ times in $\angle AOF$, and $n$ times in $\angle PHT$. Then what is the ratio of $\angle AOF$ to $\angle PHT$?

*3.* How do the small dihedral ∠s, into which the given dihedral ∠s are divided, compare? Why? How many small dihedral ∠s in dihedral $\angle MN$? How many in dihedral $\angle GK$?

*4.* What is the ratio of dihedral $\angle\ MN$ to dihedral $\angle\ GK$? Apply § 202.

*5.* How does the ratio of the plane $\angle$s compare with the ratio of the dihedral $\angle$s?

CASE II. *When the plane angles are incommensurable.*

*Suggestion.* Use the method of § 215, II., in working out the demonstration.

*Therefore —*

**448.** SCHOLIUM. Since dihedral angles are proportional to their plane angles, they are said to be measured by their plane angles. Thus, a right dihedral angle is a dihedral angle whose plane angle is a right angle, and a dihedral angle of 27° is one whose plane angle is an angle of 27°, etc.

(*a*) Compare the method of leading up to the measurement of dihedral angles with that of leading up to the measurement of plane angles by arcs.

**449.** A **plane is perpendicular** to another plane if it forms with the other plane a right dihedral angle.

**450.** Dihedral angles are adjacent, vertical, acute, obtuse, etc., when their plane angles are adjacent, vertical, acute, obtuse, etc., respectively.

---

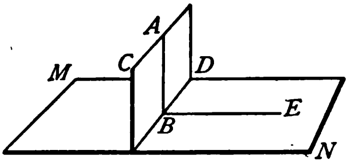
Ex. 296. If two planes intersect each other, the vertical dihedral angles are equal. §§ 446 and 448.

Ex. 297. If one plane intersects another, the sum of the two adjacent dihedral angles on the same side of either plane, is equal to two right dihedral angles.

Ex. 298. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane. Fig. Ex. 299.

## Proposition XXV.

**451. Theorem.** *If a straight line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.*




**Let A B be a straight line perpendicular to the plane M N, and let C D be any plane containing the line A B.**

*To prove that the plane C D is perpendicular to the plane M N.*

*Suggestion 1.* What must be known to prove $CD \perp$ to $MN$?

2. In the plane $MN$, erect $BE \perp$ to $BD$ the line of intersection of the two planes at point $B$.

3. How many degrees in $\angle EBA$? Why? $\angle EBA$ is the plane angle of the dihedral $\angle$, $NBDA$. Why?

4. Then, what relation does the plane $CD$ sustain to the plane $MN$? § 449.

*Therefore* —

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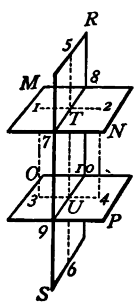
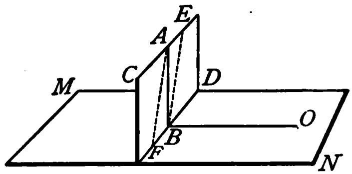



Ex. 299. If a plane intersects two parallel planes, the alternate interior dihedral angles are equal.

*Suggestion.* Pass a plane, *1 4*, $\perp$ to the edge *7 8*. What relation does this plane sustain to edge *9–10*? Compare the plane angles of the dihedrals thus formed.

## Proposition XXVI.

452. **Theorem.** *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their line of intersection is perpendicular to the other.*




**Let M N and C B be two planes perpendicular to each other, intersecting in the line B D, and let A B be the line in C D perpendicular to B D.**

*To prove that A B is perpendicular to the plane M N.*

*Suggestion 1.* What is it necessary to prove, in addition to what is given in the theorem, to know that *A B* is ⊥ to *M N*?

*2.* In the plane *M N*, draw *B O* ⊥ to *B D*, at *B*. The ∠ *A B O* is the plane ∠ of the dihedral ∠. Why? How many degrees in ∠ *A B O*? Why?

*3.* What relation does *A B* sustain to the plane *M N*? Why? § 415.

*Therefore —*

453. Corollary I. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

**Let A B be drawn perpendicular to M N at point B in B D.**

*To prove that A B lies in plane C D.*

*Suggestion 1.* Draw *E B* in plane *C D* ⊥ to *B D* at *B*. What relation does *E B* bear to *M N*? *A B* to *M N*? Give auth.

*2.* How many ⊥s can be erected to *M N* at *B*?

**454.** COROLLARY II. *If one plane is perpendicular to another, a perpendicular dropped from a point in the first plane to the second, lies wholly in the first plane.*

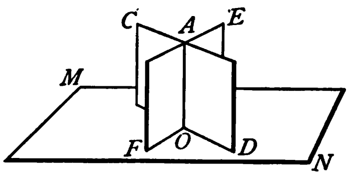
**Let A B be drawn from A in plane C D, ⊥ to M N.**

*To prove A B lies in plane C D.*

*Suggestion.* Draw *A F* in plane *C D* ⊥ to *B D* and follow plan of proof in Cor. I.

PROPOSITION XXVII.

**455. Theorem.** *If two planes are each perpendicular to a third plane, their intersection is perpendicular to that plane.*




**Let C D and E F be two planes, each perpendicular to the plane M N, and let A O be the line of intersection of C D and E F.**

*To prove that A O is perpendicular to the plane M N.*

*Suggestion 1.* From the point *O*, which is common to all three planes, erect a ⊥ to the plane *M N*.

*2.* Where does this ⊥ lie with respect to the plane *C D*? With respect to *E F*? § 453.

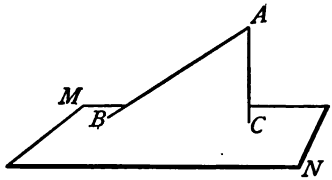
*3.* What relation does the ⊥ sustain to the intersection of the planes *C D* and *E F*? Why?

*Therefore —*

**456.** COROLLARY. *If a plane is perpendicular to the intersection of two planes it is perpendicular to the planes.* § 451.

## PROPOSITION XXVIII.

**457. Theorem.** *Through a given straight line oblique to a given plane, one, and but one, plane can be passed perpendicular to the given plane.*




**Let A B be a given straight line oblique to a given plane, M N.**

I. *To prove that one plane can be passed through A B ⊥ to M N.*

*Suggestion 1.* From any point in *A B*, as *A*, drop a ⊥ *A C* to the plane *M N*.

*2.* *A B* and *A C* determine a plane, *A B C*. Why?

*3.* What relation does the plane *A B C* sustain to the plane *M N*? § 451.

II. *To prove that but one plane can be passed through A B ⊥ to M N.*

*Suggestion 1.* If another plane can be drawn ⊥ to *M N* embracing *A B*, what relation does *A B* sustain to this plane and the plane *A B C*? § 411.

*2.* Then what relation must *A B* sustain to *M N*? § 455.

*3.* Compare this relation with the hypothesis.

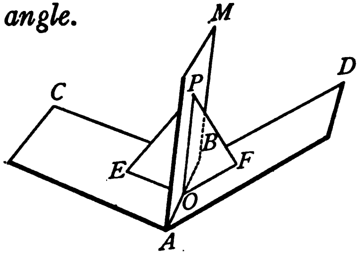
*Therefore* —

---

Ex. 300. If a plane intersects two parallel planes, the corresponding dihedral angles are equal.

## PROPOSITION XXIX.

**458. Theorem.** *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the dihedral angle.*




**Let A M be a plane bisecting the dihedral angle formed by the planes A C and A D, let P be any point in the plane A M, and P E and P F perpendiculars from P to the faces A C and A D respectively.**

*To prove that P E and P F are equal*

*Suggestion 1.* $P E$ and $P F$ determine a plane, $E F$. Let $O E$, $O P$ and $O F$ be the intersections of this plane with the planes $A C$, $A M$ and $A D$, respectively.

*2.* What relation does the plane $E F$ sustain to each of the planes $A C$ and $A D$? § 451.

*3.* What relation does the plane $E F$ sustain to $A B$, the edge of the dihedral ∠? Why?

*4.* Why are $P O E$ and $P O F$ the plane angles of the dihedrals?

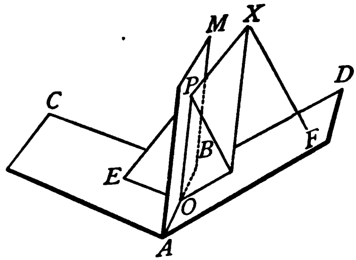
*5.* Compare ∠s $P O E$ and $P O F$, (§ 446); △s $P O E$ and $P O F$; lines $P E$ and $P F$.

*Therefore —*

QUERY: What statement in the theorem makes it necessary that $P E$ and $P F$ should be perpendicular to planes $C A$ and $A D$, respectively?

## Proposition XXX.

459. **Theorem.** *Every point out of the plane which bisects a dihedral angle is unequally distant from the faces of the dihedral angle.*



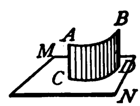

**Let A M bisect the dihedral angle C A B D, let X be any point without plane A M, and X E and X F be perpendiculars from X to planes A C and A D.**

*To prove X E and X F are unequal.*

*Suggestion.* One of the lines, as *X E*, cuts plane *M A*, as at *P*.

See method of Ex. 70.

460. The **projection of a point** upon a plane is the foot of the perpendicular from the point to the plane.




461. The **projection of a line** upon a plane is the line in the plane which contains the projections of all the points of the line.

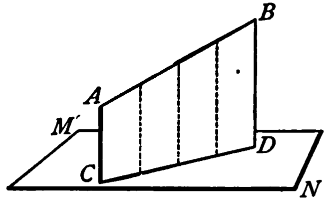
In the figure, point *C* is the projection of point *A*, and the line *C D* is the projection of line *A B*.

---

Ex. 301. Prove the converse of proposition § 446, viz. that if two dihedral angles are equal their plane angles are equal.

## Proposition XXXI.

462. **Theorem.** *The projection of a straight line upon a plane is a straight line.*




**Let A B be a given straight line and M N a given plane.**

*To prove that the projection of A B upon the plane M N is a straight line.*

*Suggestion 1.* Let a plane embracing *A B* and ⊥ to *M N* be represented by *A D*. Let *C D* represent the intersection of the planes *A D* and *M N*.

2. Where must all ⊥s dropped from *A B* to plane *M N* lie? Why? (§ 454.) In what line must the feet of the ⊥s lie? Then what kind of line is the projection?

---

Ex. 302. If any point in a plane is equally distant from two intersecting planes, the plane is the bisector of the dihedral angle formed by the two planes.

*Suggestion.* Use the indirect method.

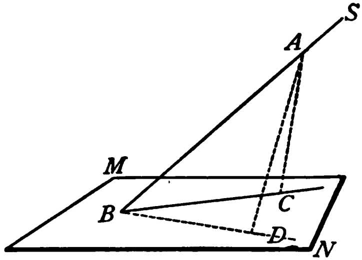
Another method. Prove *A M* bisects dihedral angle *C A B D* in figure, § 458.

*Suggestion.* Prove $\angle EOP = \angle POF$.

Ex. 303. If a plane intersects two parallel planes, the interior dihedral angles on the same side of the cutting plane are supplements of each other.

## Proposition XXXII.

**463. Theorem.** *The angle which a straight line makes with its own projection upon a plane is the least angle the line makes with any line of the plane.*




**Let S B represent any straight line, and B C its projection upon the plane M N.**

*To prove that the angle S B C is the least angle the line A B makes with any line of the plane M N.*

*Suggestion 1.* Through *B*, draw any other line of the plane, as *B D*. From any point in *S B*, as *A*, drop a ⊥ to the plane *M N*, as *A C*. Where does *C* fall? Why? Cut off *B D* equal to *B C* and connect *A* and *D*.

2. Compare *A C* and *A D* in respect to length. Give auth.

3. Compare ∠ *A B C* with ∠ *A B D*. Auth. § 139. *Therefore —*

**464.** The angle which a line makes with its projection on a plane is the **angle of the line and the plane,** or the **inclination of the line to the plane.**

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Ex. 304. What is the locus of all lines drawn through a given point parallel to a given plane?

## POLYHEDRAL ANGLES.

465. When three or more planes meet at a common point they form a **polyhedral angle.** In Fig. 1, *A - B C D* represents a polyhedral angle.

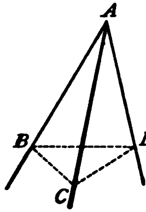



FIG. 1.

(*a*) The common point *A* is the **vertex of the angle.**

(*b*) The portions of the planes meeting at the point are the **faces** of the polyhedral angle, as *B A C, C A D,* etc.

(*c*) The intersections of the faces are the **edges** of the polyhedral, as *A B, A C,* etc.

(*d*) The plane angles formed by the edges are the **face angles** of the polyhedral, as $\angle B A C$, $\angle C A D$, etc.

(*e*) The faces of a polyhedral angle are indefinite in extent.

466. If the intersections of a plane with all the faces of a polyhedral angle form a convex polygon, the polyhedral angle is a **convex polyhedral angle.**

467. In a polyhedral angle each pair of adjacent faces forms a dihedral angle; as *B-A C-D*, etc., Fig. 1.

468. A polyhedral angle of three faces is a **trihedral angle,** one of four faces is a **tetrahedral angle.**

469. A trihedral angle is **isosceles** if two of its face angles are equal.

470. Two **polyhedral angles are equal** when they can be made to coincide.

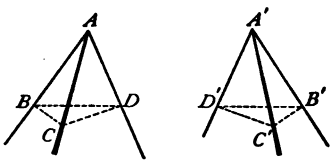



FIG. 2.

471. Two polyhedral angles are **symmetrical** when the

dihedral and face angles of one are equal respectively to the dihedral and face angles of the other, each to each, but arranged in reverse order.

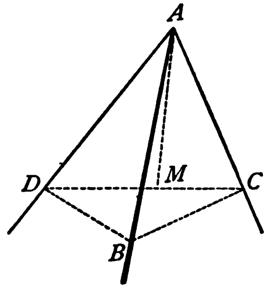
*A-B C D* and *A′-B′ C′ D′* are symmetrical if $\angle B A C = \angle B' A' C'$, $\angle C A D = \angle C' A' D'$, and $\angle B A D = \angle B' A' D'$; if dihedral $\angle A C$ = dihedral $\angle A' C'$, dihedral $\angle A B$ = dihedral $\angle A' B'$, etc., Fig. 2.

**472.** A polyhedral angle cannot be made to coincide with its symmetrical polyhedral angle except as in § 477.

NOTE.—The two hands, or the two sides of the face, of the human body will serve to illustrate symmetrical solids. The right glove cannot fit the left hand.

## PROPOSITION XXXIII.

**473. Theorem.** *The sum of any two face angles of a trihedral angle is greater than the third face angle.*




**Let A-B C D represent a trihedral angle in which each of the face angles D A B and B A C is smaller than the face angle D A C.**

*To prove that the sum of the face angles D A B and B A C is greater than the face angle D A C.*

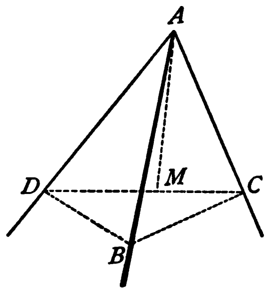
*Suggestion 1.* In the face *D A C* draw *A M*, making $\angle M A D = \angle D A B$.

2. Cut the edges by a plane $D B C$, so that $A M = A B$.

3. Compare $\Delta$s $D A B$ and $\angle M A D$; $D M$ and $D B$; $B C$ and $M C$; $\angle B A C$ and $\angle M A C$. Give auth.

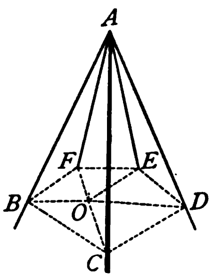
4. Compare $\angle BAC + \angle BAD$ with $\angle D A C$.

*Therefore —*




PROPOSITION XXXIV.

**474. Theorem.** *The sum of the face angles of any convex polyhedral angle is less than four right angles.*




**Let A-B C D, etc., represent a convex polyhedral angle; B A C, C A D, etc., the face angles.**

*To prove that the sum of B A C, C A D, etc., is less than four right angles.*

*Suggestion 1.* Pass a plane cutting the edges of the polyhedral $\angle$, as in the points $B$, $C$, $D$, etc.

2. Connect $O$, any point within the polygon $B D$, with each of the vertices. Compare the number of $\Delta$s whose vertices are at A with the number of $\Delta$s whose vertices are at $O$; the number of degrees in the sum of

the interior ∠s of the △s whose vertices are at $A$ with the number of degrees in the sum of the interior ∠s of the △s whose vertices are at $O$.

*3.* In trihedral $< B$ compare $\angle CBF$ with $\angle ABC + \angle ABF$ (§ 473) in the trihedral $\angle C$ compare $\angle BCD$ with $< ACB + < ACD$; in trihedral $D$ compare $< CDE$ with $< CDA + < ADE$; etc.

*4.* Compare the number of degrees in the sum of the base ∠s of the △s whose vertices are at $O$ with the number of degrees in the sum of the base ∠s of the △s whose vertices are at $A$.

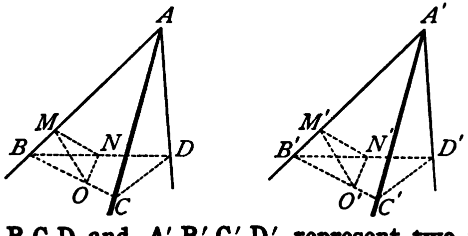
*5.* Compare the number of degrees in the vertical ∠s at $A$ with the number of degrees in the vertical ∠s at $O$. Sug. 2 and 4, and Ax. 8.

*6.* Compare the sum of the ∠s at $A$ with four rt. ∠s.

*Therefore —*

## PROPOSITION XXXV.

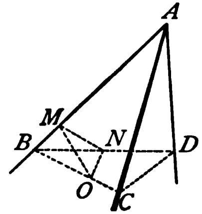
**475. Theorem.** *If two trihedral angles have the three face angles of one equal, respectively, to the three face angles of the other, the corresponding dihedral angles are equal.*




**Let A-B C D and A′-B′ C′ D′ represent two trihedral angles whose face angles are equal; viz., B A C=B′ A′ C′, C A D=C′ A′ D′, and B A D=B′ A′ D′.**

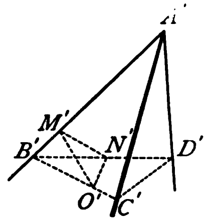
*To prove that the corresponding dihedral angles C B A D and C′ B′ A′ D′, etc., are equal.*

*Suggestion 1.* Pass planes $B C D$ and $B' C' D'$, making the edges $A B$, $A C$, $A D$, $A' B'$, $A' C'$, $A' D'$, all equal.




*2.* In $A B$ and $A' B'$ take points $M$ and $M'$ so that $A M = A' M'$, and at $M$ and $M'$ pass planes ⊥ to $A B$ and $A' B'$ respectively.

*3.* Plane $M O N$ must intersect $B C$ and $B D$, or these lines, extended, and plane $M' O' N'$, $B' C'$ and $B' D'$, or these lines extended. Why?




*4.* What relation does $< O M N$ sustain to the dihedral $\angle A B$? $O' M' N'$ to the dihedral $\angle A' B'$? Why?

The question now is how do plane $\angle$s $M$ and $M'$ of the dihedrals $A B$ and $A' B'$ compare? To discover this:

*5.* Compare $\Delta$s $A B C$ and $A' B' C'$; $<$s $A B C$ and $A' B' C'$; $B C$ and $B' C'$.

*6.* Compare $\angle$s $A B D$ and $A' B' D'$, $B D$ and $B' D'$.

*7.* Compare $\Delta$s $B M O$ and $B' M' O'$; $B O$ and $B' O'$; $M O$ and $M' O'$. In $\Delta$s $B M N$ and $B' M' N'$ compare $B N$ and $B' N'$; $M N$ and $M' N'$.

*8.* Compare $\Delta$s $D B C$ and $D' B' C'$, $\angle$s $O B N$ and $O' B' N'$; $\Delta$s $B O N$ and $B' O' N'$, $O N$ and $O' N'$.

*9.* Compare $\angle$s $M$ and $M'$.

*10.* Compare dihedrals $A B$ and $A' B'$.

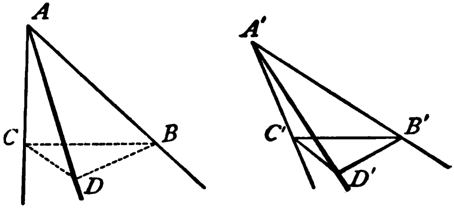
Give auth. for each step.

In the same way compare dihedrals $A C$ and $A' C'$, etc.

*Therefore —*

## PROPOSITION XXXVI.

**476. Theorem.** *If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, and arranged in the same order, the trihedral angles are equal in all respects.*




**Let A and A′ represent two trihedral angles in which ∠ B A C = B′ A′ C′, ∠ C A D = C′ A′ D′, and ∠ B A D = B′ A′ D′, etc., the angles being arranged in the same order.**

*To prove that trihedral angle A is equal to trihedral angle A′.*

*Suggestion 1.* Place thl. ∠ *A* upon thl. ∠ *A′* with line *A B* in *A′ B′* and face *A B C* lying in the plane of *A′ B′ C′*, where does line *A C* lie? Why?

*2.* Where does plane *A C D* lie (§ 475)? Where does line *A D* lie with respect to *A′ D′*? Why?

*3.* Where does plane *B A D* lie? Why?

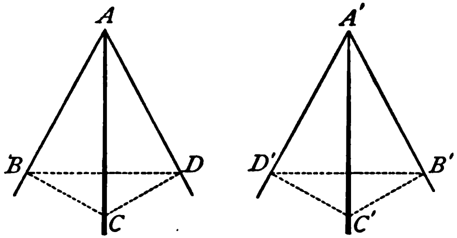
*Therefore* —

---

Ex. 305. If two planes are cut by a third plane so that the corresponding dihedral angles are equal and the edges of the dihedral angles formed are parallel, the two planes are parallel. Fig. Ex. 299.

## PROPOSITION XXXVII.

477. **Theorem.** *Two symmetrical isosceles trihedral angles are equal in all respects.*




**Let A-B C D and A′-B′ C′ D′ represent two symmetrical isosceles trihedral angles; i. e., let ∠B A C = ∠B′ A′ C′, ∠C A D = ∠C′ A′ D′, etc. Also let ∠B A C = ∠C A D, and B′ A′ C′ = C′ A′ D′.**

*To prove that the trihedral angles A-B C D and A′-B′ C′ D′ are equal in all respects.*

*Suggestion 1.* Because they are symmetrical, what face ∠s and dihedral ∠s are equal?

2. Because they are isosceles, what face angles are equal?

3. Compare face ∠ *B A C* with face < *D′ A′ C′*, face ∠ *C A D* with face ∠ *C′ A′ B′*.

4. Compare thl. ∠ *A* and thl. ∠ *A′*. § 476.

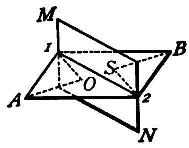
*Therefore* —

---

Ex. 306. In any trihedral angle the three planes passed through the edges, and the bisectors of the respectively opposite face angles intersect in a straight line.

Ex. 307. In any trihedral angle the three planes bisecting the dihedral angles intersect in a straight line. §§ 458 and 459.

Ex. 308. If any plane be passed through either diagonal of a parallelogram, the perpendiculars to this plane from the extremities of the other diagonal are equal.




Ex. 309. Having given a fixed straight line, and two points not in the line, find a point in the fixed line equally distant from the two fixed points.

Ex. 310. What is the locus of a point equidistant from two given parallel planes and at the same time equidistant from two given points?

Ex. 311. If two planes are cut by a third plane so that the alternate interior dihedral angles are equal and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.

Ex. 312. Find in a given plane a point which is equally distant from the vertices of a given triangle which is in another plane.

## Review.

1. When is a line parallel to a line? To a plane? A plane parallel to a plane?

2. When is a line perpendicular to a line? To a plane? A plane perpendicular to a plane?

3. When are trihedral angles equal? When symmetrical?

4. When are symmetrical trihedral angles equal in all respects?

Look carefully for all possible authorities in answering above review questions.

# CHAPTER VII.

## POLYHEDRONS.

### Definitions.

**478.** A **polyhedron** is a geometric solid bounded by planes.

(*a*) The bounding planes of a polyhedron are its **faces.**

(*b*) The lines in which the faces intersect are the **edges** of the polyhedron.

(*c*) The points in which the edges intersect are the **vertices** of the polyhedron.

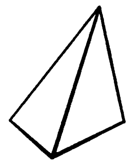


FIG. 1.

Any face designated is the **base** of a polyhedron.

**479.** A straight line joining any two vertices not in the same face is a **diagonal** of the polyhedron.

**480.** The intersection of a polyhedron and a plane is a **plane section** of the polyhedron; as Fig. 2.

**481.** Polyhedrons are classified according to the number of their faces.

(*a*) A polyhedral angle requires the meeting at its vertex of at least three planes; to completely inclose space requires at least one more plane; hence a polyhedron cannot have less than four faces.

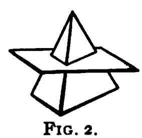


FIG. 2.

**482.** A polyhedron of four faces is a **tetrahedron;** one of five faces, a **pentahedron;** one of six faces, a **hexahedron;** one of eight faces, an **octahedron;** one of ten faces, a

**decahedron;** one of twelve faces, a **dodecahedron;** one of twenty faces, an **icosahedron,** etc.

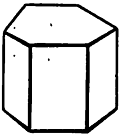


FIG. 3.

483. A convex polyhedron is one in which every possible plane section is a convex polygon.

## Prisms.

484. A **prism** is a polyhedron two of whose faces are equal polygons which lie in parallel planes, and whose remaining faces are parallelograms; as Figs. 3 and 4.

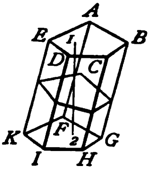



FIG. 4.

485. The **bases** of a prism are the equal polygons in the parallel planes.

(*a*) One of the bases is the **lower base** and the other is the **upper base.**

486. The **lateral faces** are the remaining faces of the prisms, as *I C*, *K D*, etc., Fig. 4.

487. The **basal edges** of a prism are the intersections of the lateral faces with the bases; as *K I*, *G H*, *E A*, etc., Fig. 4. The **lateral edges** are the intersections of the lateral faces, as *D I*, *B G*, etc., Fig. 4.

488. A **right section** of a prism is a section whose plane is perpendicular to the lateral edges of the prism, as in Fig. 4.

489. An **oblique section** of a prism is a section that is oblique to the lateral edges of the prism.

490. The **altitude** of a prism is a line that is perpendicular to the planes of its bases and intercepted between them, as 12, Fig. 4.

491. Prisms are classified as **triangular, quadrangular,** etc., according as their bases are triangles, quadrilaterals, etc.

492. A **right prism** is one in which the lateral edges are perpendicular to the bases, Fig. 5.

493. An **oblique prism** is one in which the edges are oblique to the bases, Fig. 4.

494. A **regular prism** is a right prism whose bases are regular polygons.

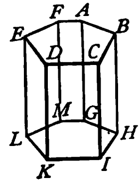
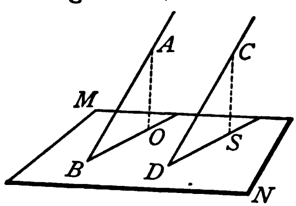



FIG. 5.

## PROPOSITION I.

495. **Theorem.** *Parallel lines intersecting the same plane make equal angles with it.*




**Let A B and C D represent two parallel lines which intersect plane M N.**

*To prove that A B and C D make equal angles with plane M N.*

*Suggestion 1.* Draw *B O* and *D S*, the projections of *A B* and *C D* respectively, upon *M N*. (Why draw these projections?)

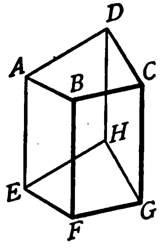
*2.* Drop ⊥s from *A* and *C*, any two points in *A B* and *C D*, to the plane. Where do the feet of these ⊥s fall? Why?

*3.* Compare ∠s *A* and *C*; *B* and *D*. Give auth.

*Therefore—*

## PROPOSITION II.

**496. Theorem.** *The lateral edges of a prism, are equal and parallel, and make equal angles with the plane of either base.*




**Let A G represent a prism and A E, B F, etc., its lateral edges.**

*To prove that A E, B F, etc., are equal and parallel, and that they make equal angles with the plane of either base.*

*Suggestion 1.* Compare *A E*, *B F*, *C G*, etc. § 433.

*2.* Compare the ∠s made by any two lateral edges with the plane of either base. Give auth.

*Therefore* —

---

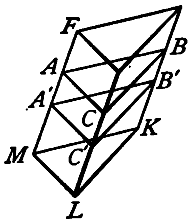
Ex. 313. If a line is parallel to each of two planes it is parallel to their intersection.

*Suggestion.* Let a plane embrace the given line and one point in the intersection of the planes. Where does it intersect each of the planes? § 428. Or apply in order §§ 428, 428, 424, 425, 428, and 424.

Ex. 314. How does a carpenter set up a studding perpendicular to the floor upon which it rests? Give geometrical authority for his work.

### PROPOSITION III.

**497. Theorem.** *Sections of a prism made by parallel planes are polygons which are equal in all respects.*




**Let A B C and A′ B′ C′ be sections of the prism F-K L M, made by parallel planes.**

*To prove that the polygons A B C and A′ B′ C′ are equal in all respects.*

*Suggestion 1.* What double relation does line *A B* sustain to *A′ B′*, *A C* to *A′ C′*, etc.? § 432.

2. Compare ∠s *A B C* and *A′ B′ C′*, also ∠s *B C A* and *B′ C′ A′*, etc. Give auth.

*Therefore —*

498. COROLLARY. *A section of a prism made by a plane parallel to a base is in all respects equal to the base.*

---

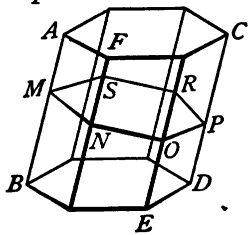
Ex. 315. Given a plane and a point without the plane, find the locus of a point in the plane at a distance, *M*, from the given point.

Ex. 316. Determine a point that is in a given plane and at equal distances from the circumference of a given circle?

Ex. 317. A point is equidistant from two points, it is also in a given plane. What is its locus?

## Proposition IV.

**499. Theorem.** *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section of the prism.*




**Let A D represent a prism, C D a lateral edge, and M P a right section of the prism.**

*To prove that the area of the lateral faces of the prism is equal to the product, C D times perimeter of M P.* § 329.

*Suggestion 1.* What relation does $P O$ sustain to $C D$? (§ 488.) What relation does any side of the right section sustain to a lateral edge which meets that side?

*2.* What is the area of the face $C E$, in terms of $P O$ and $C D$?

*3.* Express the area of each lateral face.

*4.* Express the total lateral area of the prism.

*Therefore —*

**Additional Helps.**

The number of units of area in $C E = P O \times C D$

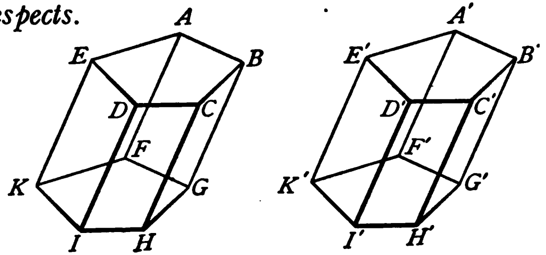
The number of units of area in $E F = N O \times C D$

The number of units of area in $F B = M N \times C D$

Number of units of lateral area $= (P O \times C D) + (N O \times C D) + (M N \times C D)$ etc. $= (P O + N O + M N$ etc.$)C D$ (factoring) $=$ Perimeter $\times$ lateral edge.

### PROPOSITION V.

**500. Theorem.** *If two prisms have the three faces of a trihedral angle of one respectively equal in all respects to the three faces of a trihedral angle of the other, and are similarly placed, the prisms are equal in all respects.*




**Let A H and A′ H′ be two prisms in which the three faces A D, A G, and A K of trihedral angle A are respectively equal in all respects to the three faces A′ D′, A′ G′, and A′ K′ of trihedral angle A′, and are similarly placed.**

*To prove that the prisms A H and A′ H′ are equal in all respects.*

*Suggestion 1.* Compare the trihedral ∠s *A* and *A′*. § 476.

2. Apply the prism *A′ H′* to the prism *A H*, so that the face *A′ D′* coincides with the face *A D*. Can this be done? (Ax. 13.)

3. In what plane does the face *A′ G′* fall? Why? In what *A′ K′*? § 475.

4. Where does the line *A′ F′* fall? Why? The point *F′*? The point *G′*? The point *K′*? The plane *G′ F′ K′*?

5. Compare faces *K G* and *K′ G′*.

Complete the superposition.

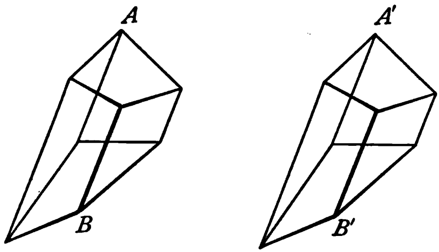
*Therefore —*

**501.** Corollary. *Two right prisms are equal in all respects if their altitudes are equal, and the bases of one are equal in all respects to the bases of the other.*

**502.** A **truncated prism** is a portion of a prism included between a base and a section of the prism made by a plane not parallel to the base.

## Proposition VI.

**503. Theorem.** *If two truncated prisms have the three faces about a trihedral angle of one respectively equal in all respects to the three faces about a trihedral angle of the other, and the faces are similarly placed, the truncated prisms are equal in all respects.*




**Let A B and A′ B′ represent two truncated prisms; let the three faces about the trihedral angle A equal respectively the three faces about A′ and be similarly placed.**

*To prove A B is equal to A′ B′.*

*Suggestion.* See the method used in § 500.

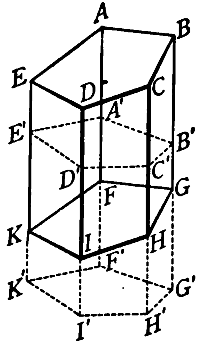
*Therefore —*

---

Ex. 318. Determine the locus of a point in space that is equally distant from the three vertices of a given triangle.

## Proposition VII.

**504. Theorem.** *An oblique prism is equal in magnitude to a right prism whose base is a right section and whose altitude is equal to a lateral edge of the oblique prism.*




**Let B K be an oblique prism, E′ B′ a right section, and E K a lateral edge.**

*To prove that the prism B K is equal in magnitude to a right prism having for a base E′ B′, and an altitude E K.*

*Suggestion 1.* Extend the lateral edge *E K* to *K′*, making *E′ K′* = *E K*. Through *K′* pass a plane *K′ G′* ⊥ to the lateral edge *E K*. Extend the other lateral edges to meet the plane *K′ G′* in the points *I′*, *H′*, etc. *E′ G′* is a right prism whose base is a right section of *B K* and whose altitude is equal to *E K*, an edge of *E G*.

*2.* *E D′* is a trapezoid. Why? Under what conditions are two trapezoids equal? Compare trapezoids *E D′* and *K I′*; *E A′* and *K F′*; polygons *E B* and *K G*. Give auth.

*3.* Compare the truncated prisms *E B′* and *K G′*.

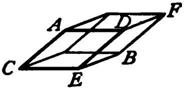
*4.* Compare prisms *E G* and *E′ G′*.

*Therefore —*

## PARALLELOPIPEDS.

**505.** A **parallelopiped** is a prism whose bases are parallelograms; as *A B*.

**506.** A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to the bases. If *A C*, *D E*, etc., are perpendicular to the bases *A F* and *C B*, *A B* is a right parallelopiped.




QUERY: If *A B* is a right parallelopiped, what kind of quadrilateral may its base be? Why?

**507.** A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles. If *A B* is a right parallelopiped, and if *A F* and *C B* are rectangles in the figure, § 506, *A B* is a rectangular parallelopiped.

**508.** A **cube** is a rectangular parallelopiped whose faces are squares.

**509.** The **volume of a polyhedron** is its ratio to some selected unit of measure times that unit. For example, if a cubic inch is contained twenty-five times in a given polyhedron the volume of the polyhedron is twenty-five cubic inches.

**510.** The **unit of measure for volume** is a cube whose edge is a given linear unit.

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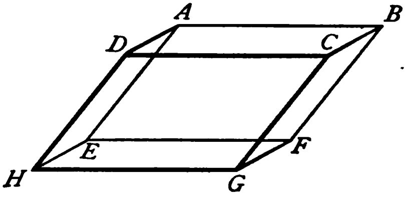
Ex. 319. Prove that the lateral area of a right prism is less than the lateral area of any oblique prism having the same base and an equal altitude.

Ex. 320. If from any point in space perpendiculars are drawn to the lateral edges of a prism, or the lateral edges extended, these perpendiculars are all in the same plane.

## Proposition VIII.

**511. Theorem.** *Opposite faces of a parallelopiped are parallelograms which are equal in all respects, and lie in parallel planes.*




**Let A G be a parallelopiped, A C and E G its bases.**

*To prove that the opposite faces, as A H and B G or A F and D G are equal in all respects.*

*Suggestion 1.* What kind of a polygon is $BG$? Is $AH$? Why? § 484.

2. Compare face $AH$ with $BG$.

3. Are $AH$ and $BG$ in ‖ planes? Why? § 430.

*Therefore —*

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Ex. 321. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.

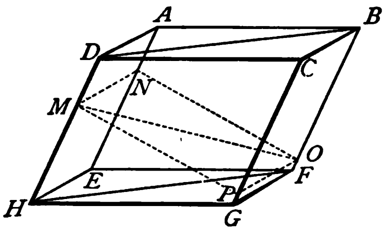
Ex. 322. If from any point in space perpendiculars are drawn to the lateral faces of a prism, or the lateral faces extended, these perpendiculars are all in the same plane.

Ex. 323. Any straight line drawn through the middle point of any diagonal of a parallelopiped, terminating in two opposite faces, is bisected at that point.

Ex. 324. The four diagonals of a rectangular parallelopiped are equal to one another.

## Proposition IX.

**512. Theorem.** *A plane passed through the diagonally opposite edges of a parallelopiped divides it into two triangular prisms which are equal in magnitude*




**Let A G be a parallelopiped, and D B F H a plane passed through the diagonally opposite edges D H and B F.**

*To prove that the triangular prisms, E F H-A and G F H-C, into which the parallelopiped is divided, are equal in volume.*

*Suggestion 1.* Pass a right section *M N O P* through the parallelopiped. What kind of quadrilateral is it?

*2.* Compare the right triangular prism whose base is *M P O* and whose altitude is a line equal to *G C*, with *M N O-E A* (base = *M N O* and alt. = *E A*). § 501.

*3.* Compare *M P O-G C* and *H G F-C*; *M N O-E A* and *H E F-A*. § 504.

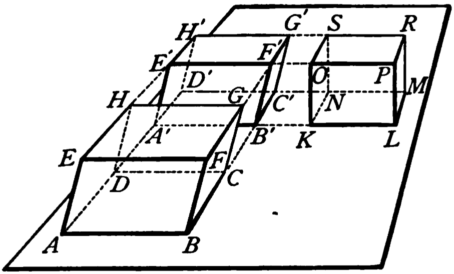
Draw conclusion.

*Therefore* —

Query: *H E F-A* and *H G F-C* have equal bases and altitudes; why cannot they be compared directly? What relation do they sustain to each other?

## PROPOSITION X.

**513. Theorem.** *Any parallelopiped is equal in volume to a rectangular parallelopiped having an equal altitude and a base equal in area.*




**Let E C represent a parallelopiped, and A C its base.**

*To prove that E C is equal in volume to a rectangular parallelopiped which has an altitude equal to the altitude of E A and a base equal in area to A C.*

*Suggestion 1.* Extend the edges *A D*, *B C*, *F G* and *E H*. Take *E' H'* = *E H*, and pass the right sections *E' B'* and *H' C'*. What kind of a parallelopiped is *E' C'* ? Apply definition.

2. Extend the edges *H' G'*, *D' C'*, *A' B'*, and *E' F'*. Take *S R* = *H' G'*, and pass the right sections *S K* and *R L*. What kind of a parallelopiped is *O M* ? § 507.

3. Compare in respect to volume *E C* and *E' C'*; *E' C'* and *O M*; *E C* and *O M*. § 504.

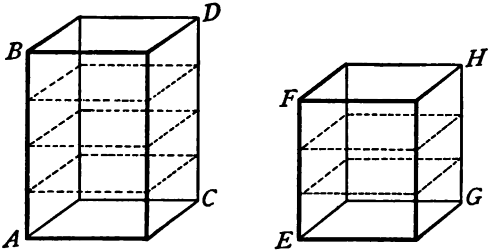
4. Compare in respect to area the bases *A C* and *A' C'*; *A' C'* and *K M*; *A C* and *K M*. § 330.

5. Compare the altitudes of *E C* and *O M*. § 433.

*Therefore* —

## PROPOSITION XI.

**514. Theorem.** *If two rectangular parallelopipeds have equal bases, the ratio of their volumes is equal to the ratio of their altitudes.*




**Let A D and E H represent two rectangular parallelopipeds whose bases A C and E G are equal, and whose altitudes are A B and E F respectively.**

*To prove that* $\frac{AD}{EH} = \frac{AB}{EF}$.

*CASE I.—*When the altitudes A B and E G are commensurable.*

*Suggestion 1.* Divide the altitudes by a common unit of measure. Let it be contained *m* times in *A B* and *n* times in *E F*. Determine the ratios of the altitudes.

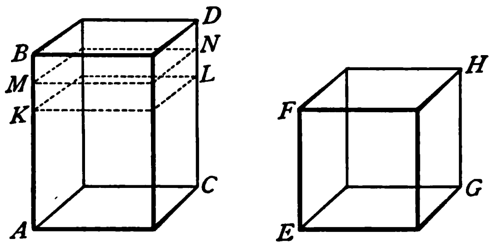
*2.* Pass planes making right sections of the parallelopipeds at the points of division. Compare in volume and number the parallelopipeds formed. Determine the ratio of the parallelopipeds.

Compare the ratio of the altitudes with the ratio of the parallelopipeds.

*Therefore* —

*NOTE.—The student should study the method in §§ 215 and 290 and test his understanding of it by trying to demonstrate both cases of this proposition without the suggestions.

CASE II.— *When the altitudes are incommensurable.*




*Suggestion 1.* If $AB$ and $EF$ are incommensurable, divide $AB$ by any unit of measure of $EF$. There will be a remainder, as $KB$, less than the unit of measure. Why?

2. Through $K$ pass a right section of the parallelopiped $AD$, as $KL$.

3. By making the unit of measure of $EF$ smaller and smaller continually, the remainder is made to decrease indefinitely. Why?

4. What relation exists between the ratios $\frac{AL}{EH}$ and $\frac{AK}{EF}$? $\frac{AN}{EH}$ and $\frac{AM}{EF}$?

5. What is the limit of $\frac{AN}{EH}$? Of $\frac{AM}{EF}$?

6. Compare the limits.

*Therefore—*

QUERY: Why is $\frac{AL}{EH}$ a variable? $\frac{AM}{EF}$? § 214 (*b*).

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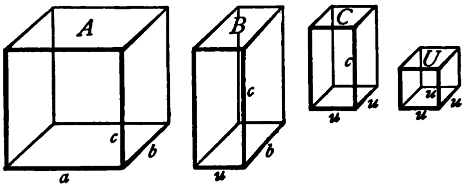
Ex. 325. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three dimensions.

Ex. 326. Find the lateral area of a regular triangular prism whose basal edges are each 4 feet and whose lateral edges are each 2 yards.

## PROPOSITION XII.

**515. Theorem.** *The number of units of volume in a rectangular parallelopiped is equal to the product of the number of linear units in the edges meeting at any vertex.*




**Let A represent any rectangular parallelopiped, and let a, b and c represent three edges meeting at a vertex. Let U be the unit of measure for volume and u, the edge of U, be the linear unit. Let $\frac{a}{u} = m$, $\frac{b}{u} = n$, and $\frac{c}{u} = r$; i.e., m represents the number of linear units in a, n in b, and r in c.**

*To prove that* $\frac{A}{U} = m \times n \times r$.

*Suggestion 1.* Construct a rectangular parallelopiped, *B*, having three edges meeting at a vertex = to *c*, *b*, and *u* respectively; a rectangular parallelopiped, *C*, having the edges at a vertex = to *c*, *u* and *u* respectively.

*2.* Determine the ratio $\frac{A}{B}$; $\frac{B}{C}$; $\frac{C}{U}$. § 514.

*3.* What then is the value of $\frac{A}{U}$? § 269.

*Therefore* —

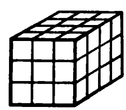
**516.** SCHOLIUM I.— In the applications of this theorem, the three edges must be expressed in terms of the *same* unit, and the unit of volume must be a cube whose edge is the linear unit. § 510.

**517.** SCHOLIUM II.— By comparison of the theorem with the definition of volume (509) it will be observed that the volume of a rectangular parallelopiped is equal to the product of the measures of three edges meeting at any vertex times the unit of measure for volume.

**518.** SCHOLIUM III.— The expression, "The product of the three dimensions," is a common abbreviation for the expression in Scholium II., "the product of the measures of three edges meeting at a vertex times the unit of measure for volume."

The product of three lines must not be interpreted in any other sense than that just stated. With this interpretation, Proposition XII. is usually stated as follows: The volume of a rectangular parallelopiped is equal to the product of its three dimensions.

**519.** SCHOLIUM IV.— When each dimension of the rectangular parallelopiped is divisible by the linear unit which is the edge of the unit of volume, the truth of the theorem may be shown by dividing the parallelopiped into cubes, each equal to the unit of measure. This method is usually employed in arithmetic.



**520.** SCHOLIUM V.— If the three edges of a rectangular parallelopiped meeting at any vertex are equal, the volume is equal to the third power of the edge and hence the third power of a number is called the cube of it.

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Ex. 327. Find the *total* area of a regular triangular prism whose basal edges are each 4 feet, and whose lateral edges are each 8 feet.

## Proposition XIII.

**521. Theorem.** *The volume of any parallelopiped is equal to the product of the area of its base by its altitude.*

*Suggestion.* § 513.

**522.** Corollary I. *If two parallelopipeds have bases which are equal in area, the ratio of their volumes is equal to the ratio of their altitudes.*

*Suggestion 1.* Let $V$, $B$, and $A$ represent respectively the volume, the area of the base, and the altitude of one parallelopiped, and $V'$, $B$, and $A'$ the volume, area of the base, and altitude of the other parallelopiped. Then by the theorem $V = B \times A$, and $V' = B \times A'$.

*2.* Then $\dfrac{V}{V'} = \dfrac{B \times A}{B \times A'} = \dfrac{A}{A'}$. § 202.

**523.** Corollary II. *If two parallelopipeds have equal altitudes, the ratio of their volumes is equal to the ratio of the areas of their bases.*

*Suggestion.* As in Cor. I., $V = B \times A$, and $V' = B' \times A$. What is the ratio $\dfrac{V}{V'}$ equal to?

---

Ex. 328. Find the lateral area of a regular pentagonal prism each edge of which is 3 inches.

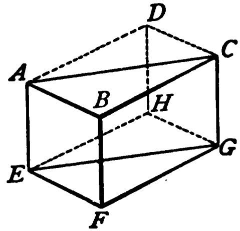
Ex. 329. Find the length of a diagonal of a rectangular parallelopiped which is 1 foot long, 9 inches wide, and 3 feet high.

Ex. 330. A rectangular tank 2′-4″ by 3′-6″ by 18′ contains how many barrels?

Indicate the solution of the problem and solve by cancellation

### PROPOSITION XIV.

**524. Theorem.** *The volume of a triangular prism is equal to the product of the area of its base by its altitude.*




**Let E F G-B represent a triangular prism and E F G its base.**

*To prove that the volume of E F G-B is equal to the product of the area of E F G by the altitude of the prism.*

*Suggestion 1.* Extend the planes of the bases. Pass a plane through the line *C G* ‖ to the face *E B*; through line *A E* ‖ to face *B G*. The figure *B H* is a parallelopiped. Why?

*2.* What is the volume of *B H*? Give auth.

*3.* Compare *E F G-B* and *B H*. Compare the bases of *E F G-B* and *B H*; the altitudes of *E F G-B* and *B H*. Give auth.

*4.* What is the volume of *E F G-B* in terms of its base and altitude?

*Therefore* —

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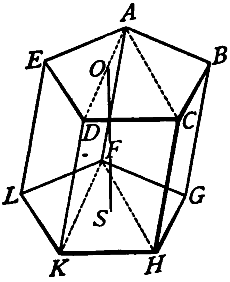
Ex. 331. Compare the volumes of two parallelopipeds whose edges are respectively 2′, 3′ and 7′, and 5′, 3′ and 8′.

Ex. 332. A parallelopiped has an altitude of 8 inches and its base is a rhombus 10 inches on a side whose shorter diagonal is 12 inches. Find its volume.

## PROPOSITION XV.

**525. Theorem.** *The volume of any prism is equal to the product of the area of its base by its altitude.*




**Let B K represent any prism, B D its base, and O S its altitude.**

*To prove that the volume of B K is equal to the product of B D by O S.*

*Suggestion 1.* Through any lateral edge, as *A F*, pass diagonal planes, *A H*, etc.

*2.* Into what kind of figures is the given prism divided by these planes?

*3.* What is the volume of *A B C-G*? of *A C D-H*? etc. Give auth. What does the sum of the bases of these prisms equal? What is their altitude?

*4.* Express the sum of these volumes in the simplest form. See § 499 for method.

*5.* What is the volume of *B K* in terms of its base and altitude?

*Therefore* —

**526.** COROLLARY I.—*If two prisms have bases which are equal in area, their volumes have the same ratio as their altitudes.*

**527.** COROLLARY II.—*If two prisms have equal altitudes, their volumes have the same ratio as the areas of their bases.*

## Pyramids.

**528.** A **pyramid** is a polyhedron all but one of whose faces meet in the same point. Fig. 1.

**529.** The point in which the faces meet is the **vertex**, as $A$.

**530.** The face which does not meet the vertex is the **base**, as $B\,C\,D$, etc.

(*a*) The vertex is always opposite the base.

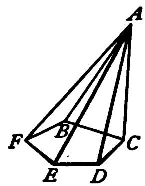


FIG. 1.

**531.** The faces which meet at the vertex are the **lateral faces**, as $A\,F\,E$, $A\,E\,D$, etc.

**532.** The edges formed by the intersections of the lateral faces are the **lateral edges**, as $A\,F$, $A\,E$, etc.

**533.** The edges formed by the intersections of the base with the lateral faces are the **basal edges**, as $F\,E$, $E\,D$, etc.

**534.** The **altitude** of a pyramid is the perpendicular from the vertex to the plane of the base.

**535.** COROLLARY.—*The altitude may be taken as the perpendicular between the base and a plane through the vertex parallel to the base. Why?*

**536.** A pyramid is **triangular, quadrangular, pentagonal,** etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.

(*a*) In a triangular pyramid any face may be taken for the base; the vertex of the opposite polyhedral angle then becomes the vertex of the pyramid.

**537.** A **regular pyramid** is one whose base is a regular polygon and whose vertex is in a perpendicular to the base erected at its center.

**538.** The **slant height** of a regular pyramid is the perpendicular from the vertex of the pyramid to any basal edge.

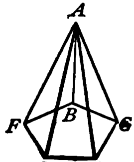


FIG. 2.

**539.** A **truncated pyramid** is the portion of a pyramid included between the base and a plane cutting all its lateral edges.

**540.** The **frustum of a pyramid** is a truncated pyramid in which the cutting plane is parallel to the base. Fig. 3. The section of the pyramid made by the cutting plane is the **upper base** of the frustum, as *H F*.

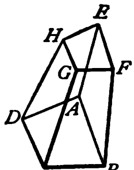


FIG. 3.

**541.** The **altitude** of the frustum of a pyramid is the perpendicular between its bases.

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Ex. 333. The sum of the squares of the four diagonals of a rectangular parallelopiped is equal to the sum of the squares of its twelve edges.

Ex. 334. Find the total area of a cube whose edge is 7 inches; 12 inches; *a* inches. Fine the volume of each cube.

Ex. 335. A rectangular box 12″ by 18″ by 22″, outside measurement, made of inch boards, contains how many cubic inches? How many cubical boxes $2\frac{3}{4}''$ on an edge can be packed in it?

Ex. 336. A cistern in the form of a regular hexagonal prism is 6′ on its basal edges and 7′ on its lateral edges. How many gallons does it hold?

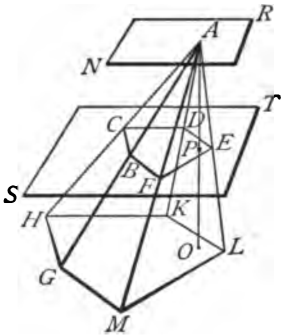
Ex. 337. The volume of a rectangular parallelopiped is 6,720 cubic inches, and its edges are in the ratio of the numbers 3, 5, and 7. Find the three edges.

## Proposition XVI.

**542. Theorem.** *If a pyramid is cut by a plane parallel to the base:*

I. *The edges and altitude are divided proportionally.*

II. *The section is a polygon similar to the base.*




**Let A-G H K, etc., represent a pyramid, B D a section made by the plane S T parallel to the base, A O the altitude, and P the point in which A O intersects the plane S T.**

I. *To prove that* $\frac{AB}{AG} = \frac{AC}{AH} = \frac{AD}{AK}$, *etc.*, $= \frac{AP}{AO}$.

*Suggestion 1.* Through $A$ pass a plane $NR$ ∥ to the plane $GL$.

2. Compare $\frac{AB}{AG}$, $\frac{AC}{AH}$, $\frac{AP}{AO}$, etc. § 437.

II. *To prove that $BD$ is similar to $GK$.*

*Suggestion 1.* What are similar polygons?

2. Establish the conditions necessary to make $BD$ and $GK$ similar polygons.

*Therefore* —

QUERY: May point $O$ be in the base extended?

**Additional Helps for II.**

1. Compare $\angle$s $BFE$ and $GML$; $FED$ and $MLK$, etc.

2. Compare $\frac{FE}{ML}$ with $\frac{AF}{AM}$ (§ 303); $\frac{BF}{GM}$ with $\frac{AF}{AM}$; $\frac{FE}{ML}$ with $\frac{BF}{GM}$, etc. Complete the conditions of the definition of similar polygons.

PROPOSITION XVII.

**543. Theorem.** *The areas of parallel sections of a pyramid are proportional to the squares of the distances of the cutting planes from the vertex.*

**In figure § 542 let C E and H L represent two parallel sections of a pyramid and A P and A O their respoctive distances from the vertex.**

*To prove* $\frac{area\ CE}{area\ HL} = \frac{\overline{AP}^2}{\overline{AO}^2}$.

*Suggestion 1.* Express the ratio $\frac{\text{area } CE}{\text{area } HL}$ in terms of two homologous sides of the polygons, as $CB$ and $HG$ (§§ 338 and 285); in terms of two lateral edges, as $AB$ and $AG$; in terms of $AP$ and $AO$.

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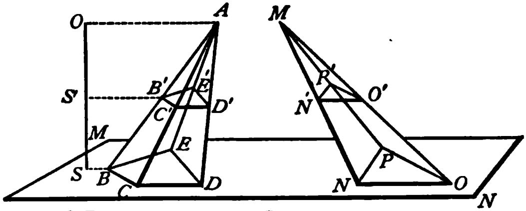
Ex. 338. Find the volume of a regular hexagonal prism whose basal edges are each 2 feet, and whose lateral edges are each 2 yards.

Ex. 339. The three edges of a rectangular parallelopiped meeting at a point are 2, 5 and 10 feet, find its lateral area and volume.

Ex. 340. A right triangular prism has an altitude of 20 inches and its basal edges are 10, 10 and 12 feet, respectively. Find its lateral area and volume.

## Proposition XVIII.

**544. Theorem.** *In pyramids whose bases are equal in area and whose altitudes are equal, sections at equal distances from the vertices and parallel to the bases are equal in area.*




**Let A-BCD, etc., and M-NOP, etc., represent two pyramids having equal altitudes, OS, and bases equal in area. Let B′C′D′ and N′O′P′ represent two sections parallel to the bases and at equal distances from A and M respectively.**

*To prove B′C′D′ and N′O′P′ are equal in area.*

*Suggestion 1.* Let $OS'$ and $OS$ represent the distance from $A$ to the section and base of pyramid $A$, respectively; also the distances from $M$ to the section and base, respectively, of the pyramid $M$. Compare ratio $\frac{B'C'D'}{BCD}$ with $\frac{N'P'O'}{NPO}$. § 543.

2. Compare $B'C'D'$ and $N'O'P'$.

*Therefore* —

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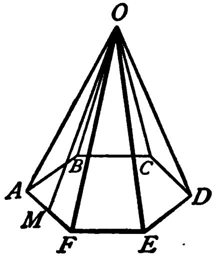
Ex. 341. Any two diagonals of a parallelopiped bisect each other.

Ex. 342. The diagonals of a parallelopiped meet in a point.

## Proposition XIX.

**545. Theorem.** *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of the base by the slant height.*



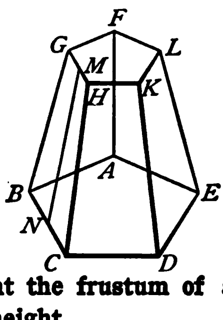

**Let O-A D represent a regular pyramid, A B D its base and O M its slant height.**

*To prove that the lateral area of the pyramid is equal to one-half the perimeter A B D by O M.*

If suggestions are needed see method § 499.

## Proposition XX.

**546. Theorem.** *The lateral area of the frustum of a regular pyramid is equal to the product of the slant height by one-half the sum of the perimeters of the two bases.*




**Let G E represent the frustum of a regular pyramid and M N its slant height.**

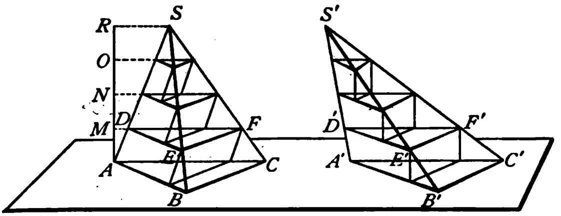
*To prove that the lateral area of the frustum G E is equal to the product of M N by one-half the sum of the perimeters of the two bases.*

If a suggestion is needed see § 499.

**547. A regular polyhedron** is a polyhedron whose faces are all equal regular polygons and whose polyhedral angles are all equal.

## Proposition XXI.

**548. Theorem.** *Two triangular pyramids having equal altitudes and bases equal in area, are equal in volume.*




**Let S-A B C and S′-A′ B′ C′ represent two triangular pyramids having equal altitudes, A R, and bases, A B C and A′ B′ C′, equal in area.**

*To prove that S-A B C and S′-A′ B′ C′ are equal in volume.*

*Suggestion 1.* Divide the altitude *A R* of the pyramids into equal parts, and through the points of division, *M*, *N*, etc., pass planes parallel to the bases, thus making sections in the pyramids. Upon each section of *S-A B C* as upper base construct a prism whose altitude is equal to the perpendicular distance between the sections and whose lateral edges are parallel to the edge $SA$ of the pyramid $S\text{-}ABC$. Similarly, construct prisms upon each section of $S'\text{-}A'B'C'$.

*2*. Compare the areas of the sections $DEF$ and $D'E'F'$. Give auth. § 544.

*3*. Compare the prisms $DEF\text{-}A$ and $D'E'F'\text{-}A'$; the two prisms adjacent to them; the next two, etc. Give auth.

*4*. Compare the sum of the prisms in $S\text{-}ABC$ with the sum of the prisms in $S'\text{-}A'B'C'$.

*5*. If the number of parts into which the altitude is divided be continually increased the sum of the prisms in $S\text{-}ABC$ is a variable; also the sum of the prisms in $S'\text{-}A'B'C'$.

*6*. Compare these variables, the number of prisms in each pyramid being the same. What is the limit of each?

*7*. Compare their limits.

*Therefore* —

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Ex. 343. Lateral faces of right prisms are rectangles.

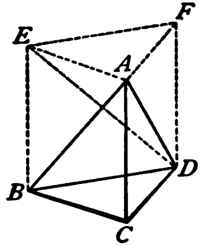
Ex. 344. Find the lateral area of a regular pentagonal pyramid whose basal edge is 2 feet, and whose slant height is 1 yard.

Ex. 345. Find the total area of a regular quadrangular pyramid whose basal edge is 8 feet, and whose lateral edge is 6 feet.

Ex. 346. The base of a regular pyramid is a square whose side is 6 feet and the lateral area of the pyramid is $\frac{5}{8}$ of its total area; find the altitude and the slant height of the pyramid.

## Proposition XXII.

**549. Theorem.** *The volume of a triangular pyramid is one-third the volume of a triangular prism having the same base and altitude.*



**Let A-B C D represent a triangular pyramid.**

*To prove that the volume of A-B C D is one-third the volume of a prism having B C D for a base and an altitude equal to the altitude of A-B C D.*

*Suggestion 1.* Through *A* pass a plane ‖ to *B C D*; extend the planes of the faces *A C D* and *B A C*; through *B D* pass a plane ‖ to the edge *A C*. § 484.

2. Why is the resulting figure a prism? Pass a plane *A E D*.

3. Compare pyds. *A-E F D* and *A-E B D*; *D-E A B* (*A-E B D*) and *D-A B C* (*A-B C D*).

4. What part of the prism is *A-B C D*? Compare its base and altitude with that of the prism.

*Therefore —*

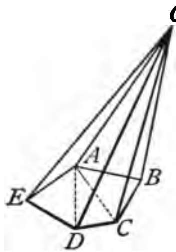
550. Corollary.— *The volume of a triangular pyramid is equal to one-third of the product of the area of its base by its altitude.*

---

Ex. 347. Find the volume of a regular triangular prism each edge of which is 4 feet.

## Proposition XXIII.

551. **Theorem.** *The volume of any pyramid is equal to one-third the product of the area of its base by its altitude.*




**Let O-E B represent a pyramid, E B its base and M its altitude.**

*To prove that the volume of O-E B is equal to one-third of the product of the area of the base E B by its altitude M.*

*Suggestion 1.* Through a lateral edge, as *O A*, pass all possible diagonal planes.

*2.* Express the volume of *O-A B C*? of *O-A C D*, etc. Give auth.

*3.* Express, in its simplest form, the sum of the volumes of the figures *O-A B C*, *O-A C D*, etc.

*4.* What is the volume of the pyramid *O-E B*?

See method § 525.

*Therefore —*

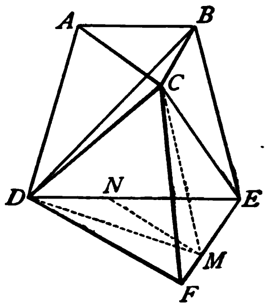
552. Corollary I.— *If two pyramids have bases which are equal in area, their volumes have the same ratio as their altitudes.*

*Suggestion.* See method § 522 or 523.

553. Corollary II.—*If two pyramids have equal altitudes, their volumes have the same ratio as the areas of their bases.*

## Proposition XXIV.

**554. Theorem.** *The volume of the frustum of a triangular pyramid is equal to the sum of the volumes of three triangular pyramids whose common altitude is the altitude of the frustum and whose bases are respectively, the upper base of the frustum, the lower base of the frustum, and a mean proportional between the bases of the frustum.*




**Let A B C-D E F represent the frustum of a triangular pyramid, A B C its upper base and D E F its lower base.**

*To prove that the volume of A B C-D E F is equal to the sum of the volumes of three pyramids, each having the same altitude as the frustum, and whose bases are respectively A B C, D E F, and a mean proportional between A B C and D E F.*

*Suggestion 1.* Through the edge *B C* and the vertex *D* pass a plane *B C D*, cutting off the pyramid *D-A C B*. Compare *D-A C B* with one pyramid of the theorem.

2. Pass the plane *C D E*, cutting off the pyramid *C-D E F*. Compare the pyramid *C-D E F* with the second pyramid of the theorem.

*3.* Draw $CM \parallel$ to $BE$, and connect $D$ with $M$. Compare the remaining pyramid, $D\text{-}CBE$, with $D\text{-}CME$. § 548.

*4.* Take $C$ as the vertex of $D\text{-}CME$ (§ 536, *a*). What is its base? Compare its altitude with that of the frustum.

*5.* To prove that the base of $C\text{-}DME$ is a mean proportional between the bases of the frustum:

(*a*) Draw $MN \parallel$ to $CA$. Compare $NM$ and $DF$.

(*b*) $\frac{\triangle DFE}{\triangle DME} = \frac{FE}{ME}$; $\frac{\triangle DME}{\triangle NME} = \frac{DE}{NE}$. Why? § 333.

(*c*) Compare the ratios $\frac{FE}{ME}$ and $\frac{DE}{NE}$; the ratios $\frac{\triangle DFE}{\triangle DME}$ and $\frac{\triangle DME}{\triangle NME}$.

(*d*) What relation does $\triangle DME$ sustain to $\triangle$s $DFE$ and $NME$?

(*e*) Compare $\triangle NME$ with $\triangle ACB$.

(*f*) Then, what relation does $\triangle DME$ sustain to the bases $DFE$ and $ACB$? Compare the pyramid $C\text{-}DME$ with the third pyramid of the theorem.

*Therefore* —

---

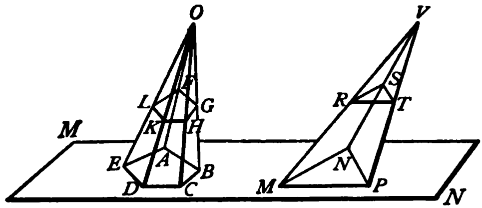
Ex. 348. Find the volume of a regular triangular pyramid whose basal edge is 4 feet, and whose altitude is $5\sqrt{3}$ feet.

Ex. 349. The point of meeting of the three medians of an equilateral triangle is $\frac{2}{3}$ of the length of the median from each vertex. Ex. 147. Ex. 261.

Ex. 350. The edges of a regular tetrahedron (§ 547) are each *a* feet. Find its slant height, altitude, base area, lateral area, total area, and volume.

## PROPOSITION XXV.

555. **Theorem.** *The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum and whose bases are respectively the upper base of the frustum, the lower base of the frustum, and a mean proportional between the bases of the frustum.*




**Let L B represent the frustum of any pyramid, L G its upper base, and E B its lower base.**

*To prove that the volume of L B is equal to the sum of the volumes of three pyramids each having the altitude of L B, and whose bases are respectively L G, E B, and a mean proportional between L G and E B.*

*Suggestion 1.* Let the lateral edges of the frustum *L B* be extended until they meet at the vertex *O*, thus forming the pyramid *O-E B*. Let *V-M N P* be a triangular pyramid whose altitude is equal to the altitude of *O-E B*, and whose base *M N P* is equal in area to the base *E B*. Let *R S T* be a section of the pyramid ‖ to the base *M N P*, and the same distance from the vertex *V* as the plane *L G* is from the vertex *O*.

2. Express volume of frustum *R S T-N M P*.

3 Compare *R S T-N M P* with frustum *L B*. What then is the volume of *L B*?

### Additional Suggestions.

Compare section $L G$ and $R S T$. § 544.

Compare pyramid $O\text{-}L G$ and $V\text{-}R S T$; pyramid $O\text{-}E B$ and $V\text{-}M N P$.

Compare altitudes and bases of the frustums.

*Therefore* —

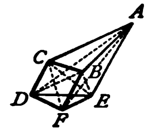
556. Corollary.— *If the volume of the frustum of a pyramid be represented by $V$, the altitude by $H$, the area of the lower base by $B$, and the area of the upper base by $b$, the truth of the theorem is expressed by the formula, $V = \frac{1}{3}H\ (B+b+\sqrt{B \times b})$. Similarly the volume of a prism is expressed by the formula $V = H B$, and the volume of the pyramid by the formula $V = \frac{1}{3} B H$.*

Note.—In solving numerical exercises in which formulæ of volume, area, etc., are involved, it is usually more convenient to indicate the operation in full and so be able to abbreviate by cancellation when possible, as: To find the volume of a frustum whose upper base is $1\frac{1}{2}$ ft. sq., the lower base 3 ft. sq., and the altitude 8 ft. Indicated, the volume is $\frac{1}{3} \times 8\ (\frac{9}{4} + 9 + \sqrt{(\frac{3}{2})^2 \times (3)^2}) = \frac{1}{3} \times 8 \times \frac{63}{4} = 42$.

---

Ex. 351. Find the volume of a regular quadrangular pyramid whose basal edge is 8′, and whose slant height is 5′.

Ex. 352. The volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids whose common base is the base of the prism and whose vertices are, respectively, the vertices of the inclined sections.




Let $A B C\text{-}D E F$ represent a truncated triangular prism.

Ex. 353. Find the volume of a truncated triangular prism, if its basal edges are 5′, 5′ and 8′, and its lateral edges 7′, 8′ and 9′, respectively.

## PROPOSITION XXVI.

**557. Theorem.** *Only five regular convex polyhedrons are possible.* § 547.

*Suggestion 1.* The regular polygon which has the smallest number of sides is the equilateral △.

*2.* How many degrees are there in an ∠ of an equilateral △?

*3.* Is it possible for three equilateral △s to meet so as to form a polyhedral ∠? (Art. 474.)

*4.* What is the least number of equilateral △s that can be used completely to enclose space?

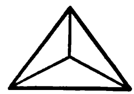


FIG. 1.

There is a regular polyhedron having four equal equilateral △s for faces. It is called a **regular tetrahedron.**

*5.* Is it possible for four equilateral △s to meet at a vertex so as to form a polyhedral ∠? Why?

*6.* How many equilateral △s are required completely to inclose space if four △s meet at each vertex?

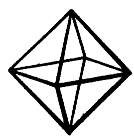


FIG. 2.

There is a regular polyhedron formed by eight equal equilateral △s. It is called a regular **octahedron.**

*7.* Is it possible for five equilateral △s to meet at a vertex so as to form a polyhedral ∠? Why?

*8.* How many equilateral △s are required completely to enclose space if five △s meet at each vertex?

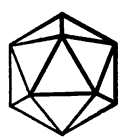


FIG. 3.

Perhaps the question in Sug. 8 cannot easily be answered without the aid of a physical figure, but by means of one constructed of cardboard, as suggested below, it is easily seen that there is a regular polyhedron formed by twenty equal equilateral △s. It is called a regular **icosahedron.**

*9.* Is it possible for six equilateral △s to meet at a vertex so as to form a polyhedral ∠ ? Why?

*10.* Are any other regular convex polyhedrons possible whose faces are equilateral △s?

*11.* How many degrees in each ∠ of a square?

*12.* Is it possible for three squares to meet at a vertex so as to form a polyhedral ∠? Why?

*13.* How many squares are required completely to enclose space if three squares meet at each vertex?

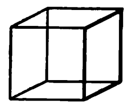


FIG. 4.

There is a regular polyhedron formed by six equal squares. It is called a regular **hexahedron** or **cube.**

*14.* Is it possible for more than three squares to meet at a vertex so as to form a polyhedral ∠ ? Why? Are any other regular convex polyhedrons possible whose faces are squares?

*15.* How many degrees in each ∠ of a regular pentagon?

*16.* Is it possible for three regular pentagons to meet at a vertex so as to form a polyhedral ∠ ? Why?

*17.* With three regular pentagons meeting at each vertex, how many are required completely to enclose space?

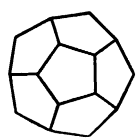


FIG. 5.

The question in Sug. 17 may be too difficult without the aid of a figure, but by means of a figure constructed of cardboard, as suggested below, it is easily seen that there is a regular convex polyhedron formed by twelve equal regular pentagons. It is called a **dodecahedron.**

*18.* Is it possible for more than three regular pentagons to meet at a vertex so as to form a polyhedral ∠ ? Why? Are any other regular convex polyhedrons possible whose faces are regular pentagons?

*19.* How many degrees in each ∠ of a regular hexagon? Is it possible for three or more regular hexagons to meet at a vertex so as to form a polyhedral ∠ ? Why? Are any regular convex polyhedrons possible whose faces are regular hexagons?

*20.* Are any regular convex polyhedrons possible whose faces are regular polygons of more than six sides?

*21.* Five regular convex polyhedrons have been enumerated; are any others possible?

*Therefore* —

558. SCHOLIUM.— The five regular polyhedrons can be constructed of cardboard, as follows: Cut the material in the shape of the following patterns, then cut it half through along the line separating the polygons. Fold it over and join the edges. Paste a strip of paper neatly over the joined edges.

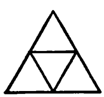


FIG. 6.

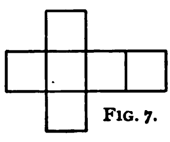


FIG. 7.

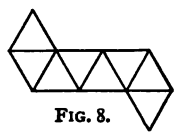


FIG. 8.

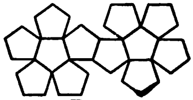


FIG. 9.

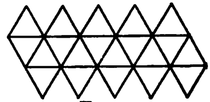


FIG. 10.

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Ex. 354. The total areas of regular tetrahedrons have the same ratio as the squares of their altitudes, or as the squares of any two homologous edges.

# CHAPTER VIII.

## THE THREE ROUND BODIES.

### The Cylinder.

**559.** A **cylindrical surface** is a surface formed by a moving straight line which always remains parallel to itself and continually touches a given curved line. The moving straight line is the **generatrix,** and the given curved line is the **directrix.**

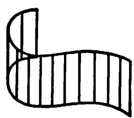


FIG. 1.

**560.** A straight line in a cylindrical surface which occupies any one of the positions of the generatrix is an **element** of the surface.

**561.** A **cylinder** is a solid bounded by a cylindrical surface and two parallel plane surfaces.

(*a*) The plane surfaces are the **bases of the cylinder,** as $A$ and $B$.

(*b*) The cylindrical surface is the **lateral surface of the cylinder.**

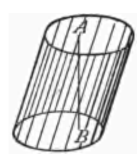



FIG. 2.

(*c*) The perpendicular between the bases is the **altitude of the cylinder,** as $A\,B$.

(*d*) COROLLARY.—*The elements of a cylinder are parallel and equal.*

**562.** A **right section of a cylinder** is a section of the cylinder that is perpendicular to an element of the surface.

**563.** A **circular cylinder** is a cylinder whose bases are circles.

**564.** The **axis of a circular cylinder** is the line that connects the centers of the bases.

**565.** A **right cylinder** is a cylinder whose elements are perpendicular to its bases.

**566.** A **cylinder of revolution** is a right circular cylinder. It can be generated by the revolution of a rectangle about one of its sides.

**567.** **Similar cylinders of revolution** are cylinders generated by the revolution of similar rectangles about homologous sides.

**568.** A **line is tangent to a cylinder** if it touches the lateral surface in a point, but does not intersect it, as *M N*. *M N* touches the surface at point *O* only.

**569.** A **plane is tangent to a cylinder** if it embraces an element of the cylinder, but does not intersect the surface, as *S P*. *S P* embraces the element 12.

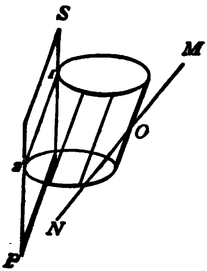


FIG. 3.

Ex. 355. If from any point in an equilateral triangle perpendiculars be drawn to the three sides, the sum of those perpendiculars is equal to the altitude of the triangle.

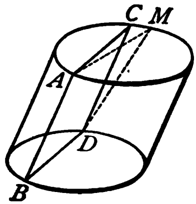
*Suggestion.* Divide the given $\triangle$ into three $\triangle$s whose common vertex is at the point from which the $\perp$s are drawn.

Ex. 356. If from any point within a regular tetrahedron perpendiculars be drawn to the four faces, the sum of these perpendiculars is equal to the altitude of the tetrahedron.

*Suggestion.* Divide the tetrahedron into four triangular pyramids whose common vertex is at the given point.

## PROPOSITION I.

**570. Theorem.** *Every section of a cylinder made by a plane embracing an element is a parallelogram.*




**Let the plane A D embrace the element A B and intersect the lateral surface in D C.**

*To prove that the section A D is a parallelogram.*

*Suggestion 1.* Let $DM$ represent the element of the cylinder through $D$. What relation does $DM$ bear to line $AB$? (§ 561 (*d*).) To plane $ABD$? To the line $DC$?

*2.* What kind of a quadrilateral is $AD$? Why?

*Therefore —*

**571.** COROLLARY. *Every section of a right cylinder embracing an element is a rectangle.*

---

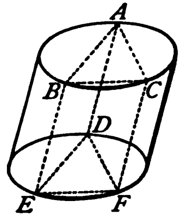
Ex. 357. If the bases of two pyramids have the same ratio as the squares of their altitudes, prove that their volumes have the same ratio as the cubes of their altitudes.

Ex. 358. What is the locus of a point in space which is at a given distance from an indefinite straight line?

Ex. 359. To cut a cylinder of revolution by a plane parallel to an element in such a manner that the section shall be a rectangle equal in all respects to the rectangle which generates the cylinder.

### Proposition II.

**572. Theorem.** *The bases of a cylinder are equal.*




**Let A E be a cylinder whose bases are A B C and D E F.**

*To prove that the bases A B C and D E F are equal.*

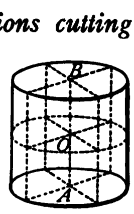
*Suggestion 1.* Take any three points in one of the bases, as *A*, *B* and *C*. Draw an element through one of them, as *A D*. Pass planes through *B* and the element *A D*; *C* and the element *A D*. Are these planes determined? These planes intersect the cylindrical surface in *B E* and *C F*. *B E* and *C F* determine a plane. Why? In what lines does this plane intersect the bases?

*2.* Compare △ *A B C* and *D E F*. § 89.

*3.* Place base *A B C* upon base *D E F*, *A* upon *D*, *B* upon *E*, etc. Why is this possible? Where does point *C* fall? How do the bases compare? Why?

*Therefore* —

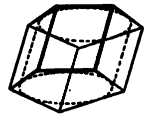
**573.** Corollary I. *Any two parallel sections cutting all the elements of a cylinder are equal.*




**574.** Corollary II. *All sections of a circular cylinder parallel to the bases are equal to the bases, and the axis passes through the centers of the sections.*

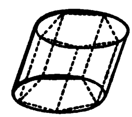
*Suggestion.* Draw two diameters of one of the bases, and through these diameters and elements of the cylinder pass planes. These planes will intersect the sections in diameters. Why?

575. A cylinder is **inscribed in a prism** when each lateral face of the prism is tangent to the cylinder, and the bases of the prism circumscribe the bases of the cylinder.



(*a*) When a cylinder is inscribed in a prism, the prism is **circumscribed about the cylinder.**

576. A **cylinder is circumscribed about a prism** when each lateral edge of the prism is an element of the cylinder, and the bases of the prism are inscribed in the bases of the cylinder.



(*a*) When a cylinder is circumscribed about a prism, the prism is **inscribed in the cylinder.**

577. COROLLARY I. *If the arcs of the base of a cylinder subtended by the basal edges of the inscribed prism are bisected, and through these points of division elements are drawn, and if planes are made to embrace these elements and the adjacent edges of the prism, an inscribed prism of a greater number of faces is formed.*

578. COROLLARY II. *If the number of faces of an inscribed prism be increased* (577) *indefinitely:*

(*a*) *The prism is a variable which approaches the cylinder as a limit.*

(*b*) *The bases of the prism are variables which approach the bases of the cylinder as limits.*

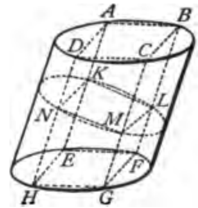
(*c*) *The lateral surface of the prism is a variable which approaches the lateral surface of the cylinder as its limit.*

---

Ex. 360. Given a right circular cylinder, required to construct the largest possible section embracing an element.

### PROPOSITION III.

**579. Theorem.** *The area of the lateral surface of a cylinder is equal to the perimeter of a right section multiplied by an element of the surface.*




**Let A G be a cylinder, K M a right section of the cylinder and A E an element.**

*To prove that the area of the lateral surface is equal to the perimeter of the section K M multiplied by A E.*

*Suggestion 1.* Inscribe a prism within the cylinder.

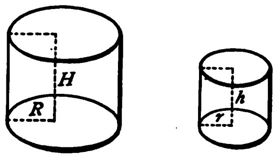
*2.* Let $L =$ the lateral area of the prism, $A$ the lateral area of the cylinder, $P$ the perimeter of a right section of the prism, $C$ the perimeter of a right section of the cylinder and $E$ an element of cylinder or edge of the prism.

*3.* $L = P \times E$. Why?

*4.* Let the number of faces be increased indefinitely. $L$ is a variable. What is its limit? Why? (§ 578 (*c*).) $P \times E$ is a variable. (§§ 371, 214 (*a*).) What is its limit? Why?

$\therefore A = C \times E$. Why?

580. COROLLARY I. *The lateral area of a cylinder of revolution is equal to the circumference of the base multiplied by the altitude. This fact can be expressed by the formula* $A = 2 \pi R H$ (§ 376), *in which A represents the area of the lateral surface, H the altitude, and R the radius of the base of the cylinder of revolution.*




581. COROLLARY II. *The lateral areas of similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.*

**Let A, H, and R represent, respectively, the area, altitude, and the radius of the base of one cylinder, and a, h, and r the area, altitude, and radius of the base of the other cylinder.**

*Then* $a = 2\pi r h$,

*and* $A = 2\pi R H$.

*Hence,* $\frac{a}{A} = \frac{2\pi r h}{2\pi R H} = \frac{r \times h}{R \times H} = \frac{r}{R} \times \frac{h}{H}$.

*But* $\frac{r}{R} = \frac{h}{H}$. *Why?* *Hence,* $\frac{a}{A} = \frac{r^2}{R^2}$, *or* $\frac{h^2}{H^2}$.

---

Ex. 361. Prove that the total surface of a right circular cylinder is equal to the circumference of its base multiplied by the sum of its altitude and the radius of its base.

Ex. 362. The total area of a right circular cylinder is 80 $\pi$ square feet and the radius of the base is 5 feet. Find the altitude of the cylinder.

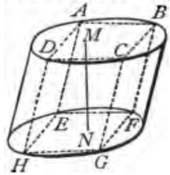
Ex. 363. Which generates the greater lateral area, the revolution of a rectangle about its shorter or longer side?

Ex. 364. A cylindrical tank on a waterworks tower has a lateral area of 1,232 square feet; the radius of its base is one-fourth of its altitude. Find the altitude and radius of the base.

Find its total area. What would its lateral area be if its altitude were twice as great? Its total area?

## Proposition IV.

582. **Theorem.** *The volume of a cylinder is equal to the area of its base multiplied by its altitude.*




**Let A G be a cylinder whose base is E G.**

*To prove that the volume of the cylinder A G is equal to the area of the base E G multiplied by the altitude.*

*Suggestion 1.* Inscribe a prism within the cylinder. Let $M$ = volume of the inscribed prism, $V$ the volume of the cylinder $A\ G$, $B$ the area of the base of the prism, $B'$ the area of the base $E\ G$ of the cylinder, and $H$ the common altitude. To prove that $V = B' \times H$.

2. $M = B \times H$. Why?

3. Let the number of the faces of the prism increase indefinitely. $M$ is a variable. What is its limit? Why? $B \times H$ is a variable. Why? What is its limit? Why?

4. What is the volume of the cylinder?

*Therefore —*

583. Corollary I. *The volume of a cylinder of revolution can be expressed by the formula $V = \pi R^2 H$, in which $V$ represents the volume, $H$ the altitude and $R$ the radius of the base of the cylinder of revolution.* § 381.

584. Corollary II. *The volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or the cubes of the radii of their bases.*

Let $v$, $h$, and $r$ represent, respectively, the volume, altitude, and radius of the base of one cylinder, and $V$, $H$, and $R$ the volume, altitude, and radius of the base of the other cylinder. § 581

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Ex. 365. The depth of a cylindrical quart cup is $4\frac{3}{4}$ inches. What is the depth of a pint cup similar in form?

**Review.**

I. Express the formulæ for:

1. The lateral area of a cylinder of revolution.
2. The total area of a cylinder of revolution.
3. The volume of a cylinder of revolution.
4. The ratio of the lateral areas of two similar cylinders of revolution.
5. The ratio of the lateral areas of any two cylinders of revolution.
6. The ratio of the volumes of two similar cylinders of revolution.
7. The ratio of the volumes of any two cylinders of revolution.

II. State the rule for finding the volume of any cylinder.

**Cones.**

**585.** A **conical surface** is a curved surface formed by a moving straight line which passes through a given fixed point and continually touches a given curve.

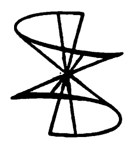


FIG. 1.

(*a*) The moving straight line is the **generatrix.**

(*b*) The given curve is the **directrix.**

**586.** A straight line in a conical surface which occupies one of the positions of the generatrix is an **element** of the surface.

587. A **cone** is a solid bounded by a conical surface and a plane surface.

(*a*) The plane is the **base of the cone.**

(*b*) The conical surface is the **lateral surface** or the **convex surface** of the cone.

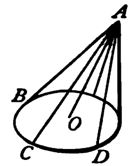


FIG. 2.

(*c*) The point in which the elements meet is the **vertex of the cone.**

(*d*) The perpendicular from the vertex to the base is the **altitude of the cone.**

588. A **circular cone** is a cone whose base is a circle. If base *B C D* in Fig. 2 is a circle, cone *A* is a circular cone.

589. A straight line joining the vertex of a circular cone with the center of the base is the **axis of the cone,** as *A O* in Fig. 2 or 3.

590. A **right cone** is a cone whose axis is perpendicular to its base.

591. COROLLARY. *The elements of a right circular cone are equal.*

*Suggestion.* Compare △s *A B O* and *A C O* in Fig. 3.

592. A right circular cone is a **cone of revolution.**

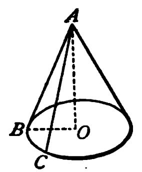


FIG. 3.

(*a*) It may be generated by revolving a right triangle about one of its legs.

(*b*) The hypotenuse in any one of its positions is the **slant height** of the cone: it is also an element of the cone.

593. **Similar cones of revolution** are cones that can be generated by the revolution of similar right triangles about homologous sides.

**594.** A **line tangent to a cone** is a line which touches the conical surface in a point but does not intersect it. $BA$, Fig. 4, is tangent to the cone $C$. $O$ is the point of tangency.

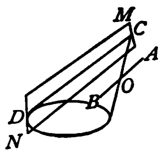


FIG. 4.

**595.** A **plane tangent to a cone** is a plane that embraces one of the elements of the cone and only one. Plane $MN$ embracing the element $CD$ is tangent to cone $C$ in Fig. 4.

**596.** A **truncated cone** is that portion of a cone included between the base and a plane which intersects all the elements.

**597.** A **frustum of a cone** is a truncated cone in which the cutting plane is parallel to the base.

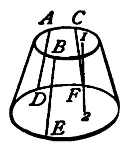


FIG. 5.

(*a*) The base of the cone is the **lower base** of the frustum, and the section parallel to the base is the **upper base.** Fig. 5.

(*b*) A line perpendicular to the bases of the frustum and intercepted by them is the **altitude of the frustum**, as 12, Fig. 5.

(*c*) The **slant height** of the frustum of a cone of revolution is that part of the slant height of the cone which is included between the bases of the frustum, as $BE$, Fig. 5.

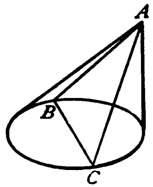
---

Ex. 366 (*a*). Which is the greater, a cylinder formed by the revolution of a rectangle about its longer side or about its shorter side?

Ex. 367. Two circular cylinders have the same altitude, but the volume of one is four times the volume of the other. Find the relation between the radii of the bases of the two cylinders.

## PROPOSITION V.

598. **Theorem.** *Every section of a cone embracing the vertex is a triangle.*




**Let A B C be a section of a cone embracing the vertex A.**

*To prove that A B C is a triangle.*

*Suggestion 1.* Draw a straight line from *A* to *B*, one point of intersection of the plane and the perimeter of the base of the cone; from *A* to the other point of intersection of the plane and the perimeter, as *C*. What are these lines?

Where is the line *A B* with reference to the plane? Why? *A C*?

*2.* Since *A B* and *A C* are both in the surface of the cone and in the plane they are the intersections of the plane and cone.

Complete the demonstration.

*Therefore* —

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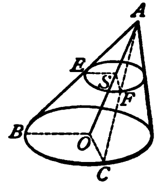
Ex. 368. In each of two right circular cylinders the altitude is equal to the diameter, and the volume of one is $\frac{1}{27}$ that of the other. Find the relation of their altitudes.

Ex. 369. A cylindrical cistern is 10 feet in diameter and 7 feet deep. Taking $7\frac{1}{2}$ gallons to the cubic foot, how many barrels does it hold?

*Suggestion.* Indicate the whole operation, then compute.

## Proposition VI.

**599. Theorem.** *Every section of a circular cone made by a plane parallel to the base is a circle.*




**Let O be the base of a circular cone A-B C, and E S F a section parallel to the base.**

*To prove that the section E S F is a circle.*

*Suggestion 1.* Let $A O$ be the axis of the cone, and $S$ the point in which the axis intersects the section $E S F$. Let $E$ and $F$ be any two points whatever in the perimeter of the section $E S F$.

*2.* Through the axis $A O$ and the point $E$ pass a plane intersecting the base in the line $O B$ and the section in the line $S E$. Also through the axis $A O$ and the point $F$ pass a plane intersecting the base in the line $O C$ and the section in the line $S F$.

*3.* What relation does $S E$ bear to $B O$? $S F$ to $O C$?

*4.* Compare ratio $\frac{E S}{B O}$ with $\frac{S F}{O C}$. § 432.

*5.* Compare $E S$ and $S F$.

*Therefore —*

*Additional Suggestions:*

Compare $\frac{E S}{B O}$ with $\frac{A S}{A O}$; $\frac{S F}{O C}$ with $\frac{A S}{A O}$; $\frac{E S}{B O}$ with $\frac{S F}{O C}$.

**600.** COROLLARY. *The axis of a circular cone passes through the centers of all sections parallel to the base.*

**601.** A cone is **inscribed in a pyramid** when the vertex of the cone coincides with the vertex of the pyramid and the base of the cone is inscribed in the base of the pyramid.

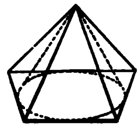


FIG. 6.

(*a*) When a cone is inscribed in a pyramid, the pyramid is **circumscribed about the cone.**

**602.** A cone is **circumscribed about a pyramid** when the vertex of the cone coincides with the vertex of the pyramid and the base of the cone is circumscribed about the base of the pyramid.

(*a*) When a cone is circumscribed about a pyramid the pyramid is **inscribed in the cone.**

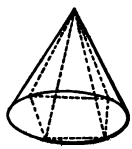


FIG. 7.

**603.** COROLLARY I. *If the arcs subtended by the basal edges of an inscribed pyramid be bisected and through the points of bisection elements of the cone be drawn, and if planes be made to embrace these elements and the adjacent edges of the pyramid, an inscribed pyramid of a greater number of faces is formed.*

**604.** COROLLARY II. *If the number of the faces of an inscribed pyramid be made to increase indefinitely:*

(*a*) *The pyramid is a variable which has for its limit the cone.*

(*b*) *The base of the pyramid is a variable which has for its limit the base of the cone.*

(*c*) *The lateral area of the pyramid is a variable which has for its limit the lateral area of the cone.*

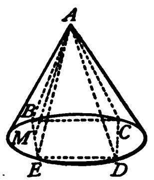
(*d*) *If the inscribed pyramid is regular, the slant height is a variable which approaches an element of the cone as a limit.* § 414.

605. COROLLARY III-IV. *Formulate for corollaries III and IV statements similar to corollaries I and II concerning the pyramids circumscribed about a cone.*

*Compare the slant height of the inscribed pyramid with that of the circumscribed pyramid.*

PROPOSITION VII.

606. **Theorem.** *The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base by its slant height.*




**Let A-B D be a cone of revolution, B E D the circumference of its base and A M its slant height.**

*To prove that the lateral area of the cone is equal to one-half the product of B E D by A M.*

*Suggestion 1.* Inscribe a regular pyramid in the cone.

2. What is the lateral area of the pyramid equal to?

Complete the demonstration by a method similar to that of § 579. § 214 (*c*).

Another method. Use a pyramid circumscribed about the cone and prove the proposition. Which authority in § 214 should be used?

*Therefore —*

**607.** COROLLARY I. *The truth of the theorem can be expressed by the formula* $A = \pi R S$; *in which A represents the lateral area, R the radius of the base, and S the slant height of the cone of revolution.*

**608.** COROLLARY II. *The lateral areas of two similar cones of revolution are to each other as the squares of their slant heights, the squares of their altitudes or the squares of the radii of their bases.*

See method of § 581.

**609.** COROLLARY III. *The lateral area of a frustum of a cone of revolution is equal to one-half the product of its slant height by the sum of the perimeters of its bases.*

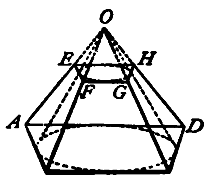


FIG. 8.

*Suggestion.* Circumscribe the frustum of a regular pyramid about the frustum of the cone and use the method of the proposition. Let $r$ be the radius of the upper, $R$ the radius of the lower base and $S$ the slant height. Prove that the lateral area of the frustum may be expressed as $\pi S (r + R)$.

**610.** COROLLARY IV. *Let M represent the radius of a section of the frustum parallel to the bases and midway between them. Prove that the lateral area of the frustum of a cone of revolution is equal to* $2 \pi S M$.

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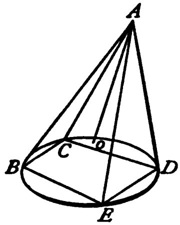
Ex. 370. The volume of the frustum of any cone is equal to one-third the altitude multiplied by the sum of the upper base, the lower base, and a mean proportional between the two bases of the frustum. § 555.

Ex. 371. How many square yards of cloth in a conical tent $12\frac{1}{2}$ feet high the diameter of whose base is 12 feet?

PROPOSITION VIII.

**611. Theorem.** *The volume of a cone is equal to one-third the product of the area of its base by its altitude.*




**Let A-B D represent a cone, A O its altitude, and B D its base.**

*To prove that the volume of the cone is equal to one-third the product of A O by B D.*

*Suggestion 1.* Inscribe a pyramid in the cone.

*2.* Express the volume of the pyramid, in terms of base and altitude.

*3.* Complete the demonstration by a method similar to that of § 582.

*Therefore —*

**612.** COROLLARY I. *The volume of a circular cone can be expressed by the formula*

$$V = \frac{1}{3} \pi R^2 H,$$

*in which V represents the volume, H the altitude, and R the radius of the base of the cone.*

**613.** COROLLARY II. *The volumes of similar cones of revolution are to each other as the cubes of the radii of their bases, the cubes of the altitudes, or the cubes of the slant heights.*

*Suggestion.* For method, see § 581.

**Review:**

**614.** State formulæ for:

1. The area of a circle.
2. The lateral area of a cone of revolution.
3. The lateral area of the frustum of a cone of revolution.
4. The volume of a circular cone.
5. The ratio of the volumes of two similar cones of revolution.
6. The ratio of the lateral areas of two similar cones of revolution.

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Ex. 372. Find the volume of the solid generated by revolving an equilateral triangle whose side is 6 feet about one of its sides.

Ex. 373. The altitude of the frustum of a cone of revolution is $\frac{1}{3}$ the altitude of the cone. Find the relation between the volume of the frustum and the cone.

Ex. 374. What is the volume of a piece of timber 15 feet long, whose bases are squares, each side of one base being 14 inches, and each side of the other base 12 inches?

Ex. 375. If four similar cylinders of revolution have their altitudes proportional to the numbers 3, 4, 5, and 6, prove that the volume of the largest cylinder is equal to the sum of the volumes of the other three.

Ex. 376. Prove Proposition § 611 by circumscribing a pyramid about the cone.

Ex. 377. What is the locus of a point at a given distance from a given point (1) in a plane (Ex. 100)? (2) In space?

## The Sphere.

**615.** A **sphere** is a solid bounded by a surface all points of which are equally distant from a fixed point within.

(*a*) The fixed point is the **center** of the sphere, as *O*, Fig. 1.

(*b*) The bounding surface is the **surface** of the sphere.

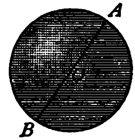


FIG. 1.

**616.** COROLLARY. *A sphere can be generated by the revolution of a semi-circle about its diameter.*

**617.** The **radius** of a sphere is any straight line drawn from the center to the surface of the sphere, as *A O*, Fig. 1.

**618.** The **diameter** of a sphere is any straight line drawn through the center and terminated by the surface, as *A O B*, Fig. 1.

**619.** COROLLARY I. *All radii of a sphere or equal spheres are equal.*

**620.** COROLLARY II. *All diameters of a sphere or equal spheres are equal.*

**621.** A **line is tangent to a sphere** if it touches the surface of the sphere at one and only one point, as *A B*, Fig. 2.

**622.** A **plane is tangent to a sphere** if it touches the sphere at one and only one point, as *M N*, Fig. 2.

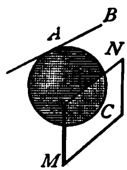


FIG. 2.

**623.** Two **spheres are tangent to each other** if they touch at one and only one point, as *O* and *S*, Fig. 3.

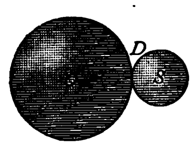
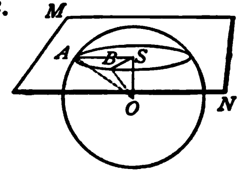


FIG. 3.

**624.** The point at which the line, plane, or sphere touches the sphere is the **point of tangency,** as *A* or *C*, Fig. 2, or *D*, Fig. 3, respectively.

PROPOSITION IX.

**625. Theorem.** *Every section of a sphere made by a plane is a circle.*




**Let the plane M N intersect the sphere O in the section A S B.**

*To prove that the section A S B is a circle.*

*Suggestion 1.* From the center of the sphere $O$ drop a ⊥ $O S$ to the plane, and from $S$ draw $S A$ and $S B$ to any two points in the perimeter of the section. Connect $A$ and $O$, $B$ and $O$.

2. Compare △s $A O S$ and $B O S$, lines $S A$ and $S B$. Why is the section $A S B$ a ⊙?

*Therefore —*

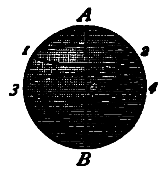


FIG. 4.

626. The section of a sphere made by a plane is a **circle of a sphere.**

627. A **great circle of a sphere** is a circle of the sphere that contains the center of the sphere, as 34, Fig. 4.

628. A **small circle of a sphere** is a circle of the sphere that does not contain the center of the sphere, as 12, Fig. 4.

629. A diameter of a sphere perpendicular to a circle of the sphere is the **axis of the circle of the sphere,** as $A B$ Fig. 4.

630. COROLLARY I. *The line drawn from the center of a sphere to the center of a circle of the sphere is perpendicular to the circle of the sphere.*

**631.** Corollary II. *If two circles of a sphere are equally distant from the center, they are equal.*

**632.** Corollary III. *Of two circles of a sphere unequally distant from the center, that one is greater which is nearer the center.*

**633.** Corollary IV. *All great circles of a sphere are equal.*

**634.** Corollary V. *The center of a great circle lies at the center of the sphere.*

**635.** Corollary VI. *Two great circles of the same sphere intersect in their diameters.*

**636.** Corollary VII. *Two great circles of the same sphere bisect each other, the sphere and the surface of the sphere.*

**637.** The **poles of a circle of a sphere** are the extremities of the axis of the circle of the sphere, as points $A$ and $B$, Fig. 4.

### Proposition X.

**638. Theorem.** *Three points, on the surface of a sphere, determine a circle of the sphere.*

*Suggestion 1.* How many points determine a plane? Give auth.

*2.* What relation must the plane bear to the sphere? Give auth.

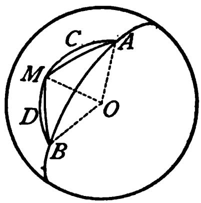
*Therefore —*

**639.** Corollary. *Two points on the surface of a sphere determine a great circle of a sphere if the points are not the extremities of a diameter of the sphere.*

*Suggestion.* What third point of the plane of the circle is known in addition to the two given points?

## Proposition XI.

**640. Theorem.** *The shortest line between any two points on the surface of a sphere is the arc of a great circle of the sphere which joins them, if that arc is less than a semi-circumference.*




**Let A and B be two points on the surface of a sphere, let A B be the arc of a great circle, not greater than a semi-circumference, and let A C M D B be any other line on the surface of the sphere joining A and B.**

*To prove that A B is less than A C M D B.*

*Suggestion 1.* Connect any point of *A C M D B*, as *M*, with *A* and *B* by arcs of great ⊙s, as *M A* and *M B*. Connect *A*, *M* and *B* with *O*, the center of the sphere. *2.* In the trihedral $\angle O - AMB$ compare the face $\angle AOB$ with the sum of the face $\angle$s *A O M* and *M O B*.

*3.* Compare arc *A B* with the sum of arcs *A M* and *M B*.

*4.* In the same way connect *C*, any point in the line *A C M*, with *A* and *M* by arcs of great ⊙s. Connect *D* in the line *M D B* with *M* and *B* by arcs of great ⊙s.

*5.* Compare the sum of arcs *A C*, *C M*, *M D*, and *D B* with the sum of arcs *A M* and *M B*; with *A B*.

*6.* Continue to take points in the line *A C M D B* and proceed as before. The sum of the arcs is a variable. Why? What is its limit? Why?

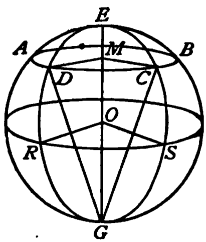
*7*. How does each succeeding value of the variable compare with the one preceding it?

*8*. Then, how must *A C M D B* compare with *A B*? *Therefore*—

**641.** The **distance between two points on the surface of a sphere** is the distance measured on the arc of a great circle.

## PROPOSITION XII.

**642. Theorem.** *All points in the circumference of a circle of a sphere are equally distant from either of its poles.*




**Let A B C D be the circumference of a circle of a sphere and E and G its poles.**

*To prove that all points in the circumference A B C D are equally distant from E and G, respectively.*

*Suggestion 1*. Let *M* be the center of the ⊙ *A B C D*. The straight line passing through *O* and *M*, if extended, passes through *E* and *G*, the poles of the ⊙ *A B C D*. Why? §§ 629 and 637.

*2*. The distances of all points in the circumference *A B C D* from any point in *E G* are how related? Why? § 414. Compare distances from *E* to circumference *A D C B*; from *G* to *A D C B*.

*3.* But the distances considered in Sug. 2 are chords of great ⊙s. Why?

*4.* How, then, must the distances measured on the surface of the sphere compare? Give auth.

*Therefore*—

**643.** The distance measured on the surface of a sphere from any point in the circumference of a circle of a sphere to its nearer pole is the **polar distance of the circle.**

**644.** COROLLARY I. *The polar distance of a great circle is a quadrant, i. e., an arc of ninety degrees.*

*Suggestion.* What is the ∠ at the center of a sphere that subtends the polar distance of a great circle?

**645.** COROLLARY II. *A point which is at the distance of a quadrant from each of two points on the surface of a sphere, is a pole of a great circle embracing those points.*

*Suggestion.* Let *E*, § 642, be the distance of a quadrant from both *R* and *S*. *R* and *S* determine what?

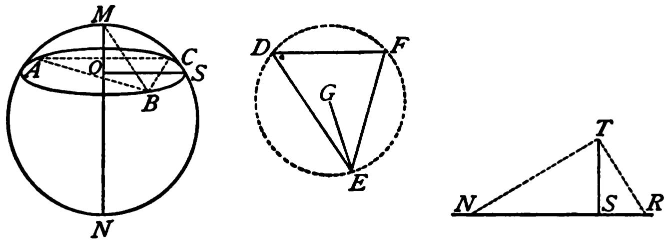
Then ∠ *R O E* is = to how many degrees? *S O E*? What relation does *E G* sustain to the circle *R O S*?

**646.** SCHOLIUM. By means of poles, arcs of circles can be drawn on the surface of a sphere in much the same way as upon a plane surface. By revolving the arc *E A* about *E* as a pole, the point *A* describes the circumference *A D C B*. By revolving the quadrant *E R* about *E* as a pole, or the quadrant *G R* about *G* as a pole, the point *R* describes the circumference of the great circle *R S*. Fig. § 642.

(*a*) To describe a great circle arc on a sphere by means of dividers, open the dividers so that their extremities touch points on the sphere which are just ninety degrees apart, then with the extremity of one leg of the dividers fixed at some point on the sphere, the extremity of the other leg describes the arc of a great circle.

## Proposition XIII.

**647. Problem.** *Given a material sphere, to find its radius.*




*Suggestion 1.* Take any point $M$ on the surface of the sphere as a pole, and describe a circumference $A\ B\ S\ C$ on the surface.

*2.* Take any three points $A$, $B$, and $C$, on this circumference, and by means of dividers construct a $\triangle\ D\ E\ F$, equal in all respects to $\triangle\ A\ B\ C$. § 89.

*3.* Circumscribe a ⊙ about the $\triangle\ D\ E\ F$, and let $G$ be the center of this ⊙.

*4.* Draw $S\ T$ equal to the radius $G\ E$, and through $S$ draw an indefinite line $N\ R \perp$ to $T\ S$.

*5.* From $T$ lay off $T\ R =$ to chord $M\ B$.

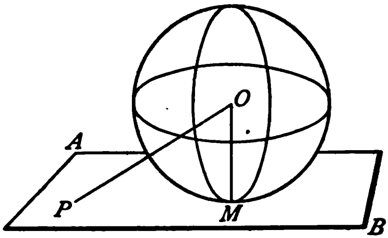
*6.* At $T$ erect a $\perp$ to $T\ R$, and extend to $N\ R$, as at $N$.

*7.* Prove that $N\ R$ is equal to the diameter of the sphere. Find the radius of the sphere.

Queries: Given the radius of a sphere. How may a quadrant be obtained? What is the value of knowing the diameter of any given material sphere, for instance, a blackboard globe? § 646.

### PROPOSITION XIV.

**648. Theorem.** *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*




**Let A B represent a plane perpendicular to the radius O M at M.**

*To prove that the plane A B is tangent to the sphere O.*

*Suggestion 1.* Draw any other line, as $O\ P$, from $O$ to the plane $A\ B$.

*2.* Compare $O\ P$ and $O\ M$ in respect to length. Give auth. § 408.

*3.* Where, then, must the point $P$ lie with respect to the sphere?

*4.* What relation does the plane $A\ B$ sustain to the sphere $O$?

*Therefore—*

**649.** COROLLARY I. *A plane tangent to a sphere is perpendicular to the radius at the point of tangency.*

*Prove that the tangent plane $A\ B$ is $\perp$ to $M\ O$.* Use Fig. § 648.

*Suggestion.* Compare $P\ O$ and $M\ O$.

**650.** COROLLARY II. *Any straight line through the point of tangency in a tangent plane is tangent to the sphere.*

*Prove $P\,M$ tangent to the sphere.* Fig. § 648.

**651.** Corollary III. *Any straight line perpendicular to the radius of a sphere at its extremity is tangent to the sphere.*

**652.** The **angle of two arcs** is the angle of two straight lines tangent, respectively, to the two arcs at the point of their intersection, as the spherical angle $M\ O\ N$ is equal to the plane angle $2\ O\ 1$, if $O\ 2$ is tangent to the curve $O\ M$ at $O$, and $O\ 1$ is tangent to the curve $O\ N$ at $O$.

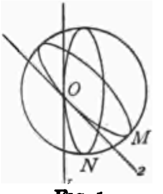



Fig. 5.

**653.** A **spherical angle** is an angle formed by the arcs of two great circles.

**654.** Corollary. *A spherical angle is equal to the dihedral angle formed by the planes of the circles whose circumferences form the spherical angle.*

*Suggestion.* The tangents are in the planes, respectively, of the ⊙s and ⊥ to the edge of the dihedral ∠. Why? § 190.

---

Ex. 378. If one circle of a sphere passes through the poles of another circle of a sphere, the planes of the two circles are perpendicular to each other.

Ex. 379. All lines tangent to a sphere from the same point are equal and touch the sphere in the circumference of a circle of the sphere.

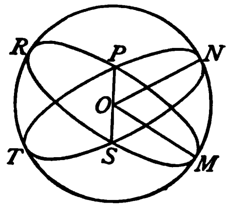
*Suggestion.* Connect the center of the sphere with the given point and with two or more points of tangency.

Ex. 380. The surfaces of two spheres intersect in the circumference of a circle. The spheres intersect in a circle.

Ex. 381. What is the locus of a point in space at a given distance from a given point?

## PROPOSITION XV.

**655. Theorem.** *A spherical angle is measured by the arc of a great circle described from the intersection of the arcs as a pole and intercepted between them.*




**Let M S and N S represent two arcs of great circles intersecting at S, and M N the arc of a great circle intercepted between the arcs S N and S M, whose pole is at S.**

*To prove that the spherical angle M S N is measured by the arc M N.*

*Suggestion 1.* Let *O N* and *O M* be radii of the great ⊙ *M N R T* drawn, respectively, in the planes of the great ⊙s *P M S* and *P N S*. What relation do *O N* and *O M* bear to *P S*?

2. What is the relation of ∠ *M O N* to the dihedral ∠ *M-S P-N*? Why?

3. What, therefore, is the relation of ∠ *M O N* to the spherical ∠ *M S N*?

4. What is the relation of arc *M N* to ∠ *M O N*? What to spherical ∠ *M S N*?

*Therefore—*

**656.** A **spherical polygon** is a portion of the surface of a sphere bounded by arcs of great circles, as *A B C D*.

(*a*) The bounding arcs, *A B*, *B C*, etc., are the **sides of the polygon.** Fig. 6.

(*b*) The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is at the center of the sphere. Thus, *O-A B C D* is a polyhedral angle whose vertex is at *O*, the center of the sphere.

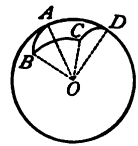


FIG. 6.

(*c*) As the sides of a spherical polygon are arcs of great circles, they are expressed in degrees and are the measures of the face angles of the polyhedral angle. Arc *B C* is the measure of face angle *B O C*. Fig. 6.

(*d*) The diagonal of a spherical polygon is the arc of a great circle joining any two vertices of the polygon not adjacent.

657. A **convex spherical polygon** is a spherical polygon none of whose sides if extended would cut the polygon, as *A B C D*, Fig. 6.

658. A **concave spherical polygon** is a spherical polygon, two or more of whose sides if extended would cut the polygon. *E D* and *C D*, Fig. 7, if extended, would cut the polygon.

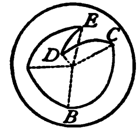


FIG. 7.

659. Spherical triangles are right angled, isosceles, equilateral, etc., in the same relations as are plane triangles.

660. Arcs of great circles on the same sphere or equal spheres can be superposed in the same way as straight lines are superposed upon a plane surface.

661. **Equal spherical angles** can be applied to each other and made to coincide in the same manner as equal plane angles.

662. **Equal spherical triangles** are spherical triangles, which, by placing one upon the other, can be made to coincide in every part. In such triangles each side and

angle of one is equal to the corresponding side and angle of the other, respectively.

**663. Symmetrical spherical triangles** are spherical triangles in which the sides and angles of one are equal, respectively, to those of the other, but arranged in reverse order. Such triangles cannot, except in the case stated in § 664, be placed one upon another so as to coincide.

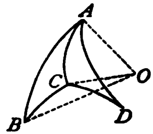
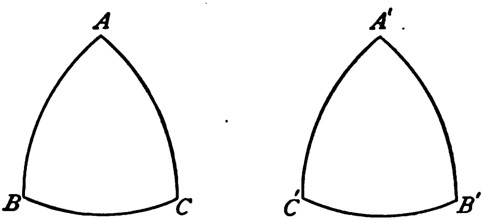



FIG. 8.

## PROPOSITION XVI.

**664. Theorem.** *Two isosceles symmetrical spherical triangles can be made to coincide, and are equal.*




**Let A B C and A′ B′ C′ represent two symmetrical spherical triangles, i. e., let A B=A′ B′, A C=A′ C′, B C=B′ C′, ∠ A=∠ A′, ∠ B=∠ B′, and ∠ C=∠ C′; let A B=A C and A′ B′ =A′ C′.**

*To prove that the triangles A B C and A′ B′ C′ are equal.*

*Suggestion 1.* Compare *A′ C′* with *A. B*; *A′ B′* with *A C*. Give auth.

2. Place *A′ B′ C′* upon *A B C* so that *A′ C′* falls upon *A B*, *A′* upon *A*. Where does *C′* fall? Where does

$A'$ $B'$ fall? Why? Where does $B'$ fall? Why? Where does arc $C'$ $B'$ lie?

*Therefore*—

**Compare with the demonstration § 82.**

QUERY: Connect the vertices of $\triangle$s $A$ $B$ $C$ and $A'$ $B'$ $C'$ with the centers of their respective spheres. What kind of trihedral angles are formed? How do they compare?

**665.** If from the vertices of a spherical triangle as poles, arcs of great circles be drawn, a second spherical triangle is formed which is the **polar** of the first triangle. If from $A$, $B$, and $C$ as poles, arcs of great circles be drawn, a triangle $D$ $E$ $F$ is formed which is the polar of the triangle $A$ $B$ $C$.

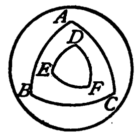


FIG. 9.

If the entire circles be drawn they will intersect to form eight spherical triangles, but the polar of the given triangle $A$ $B$ $C$ is that one of the eight triangles whose vertices lie on the same side of the arcs of the given triangle as the respectively homologous vertices of the given triangle, and no side of which is greater than 180 degrees.

Represent on a spherical blackboard or other sphere the eight spherical triangles and distinguish a triangle and its polar.

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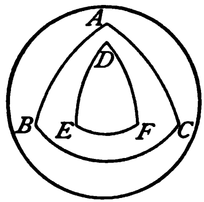
Ex. 382. What is the locus of a point that is a given distance from a given point and also a given distance from another point? Discuss the possibilities of the solution under various distances and locations of the given points.

## PROPOSITION XVII.

**666. Theorem.** *If a spherical triangle is the polar of another spherical triangle, then the second triangle is the polar of the first.*




**Let the spherical triangle A B C be the polar of D E F.**

*To prove that triangle D E F is the polar of triangle A B C.*

*Suggestion 1.* What must be proved concerning $A$, $B$ and $C$ to know that $\triangle\ D\ E\ F$ is the polar of $\triangle\ A\ B\ C$?

*2.* See § 645 and establish that $C$ is the pole of arc $D\ E$; that $A$ is the pole of arc $F\ E$, etc.

Complete the demonstration.

*Therefore—*

**667.** When two spherical triangles are each the polar of the other, they are called **polar triangles.**

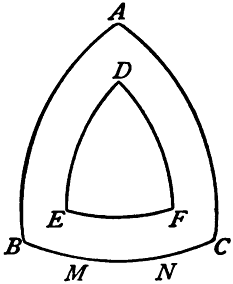
---

Ex. 383. A straight line tangent to a circle of a sphere lies in a plane which is tangent to the sphere at the point of contact.

Ex. 384. Compute the lateral area, the total area, and the volume of the frustum of a cone of revolution, given the altitude of the frustum 20 feet, the diameter of the lower base 16 feet and the diameter of the upper base 12 feet.

## Proposition XVIII.

**668. Theorem.** *In two polar triangles each angle of one is measured by the supplement of the side opposite it in the other.*




**Let A B C and D E F be two polar triangles in which side B C of triangle A B C is opposite angle D of triangle D E F, etc.**

*To prove that the angle D is measured by the supplement of side B C.*

*Suggestion 1.* Define supplement of an angle or arc.

*2.* How is a spherical angle measured? § 655.

Hence, extend arcs $DE$ and $DF$ to meet arc $BC$ at $M$ and $N$. Which arc is the measure of $\angle D$?

*3.* To establish that this arc is the supplement of $B\,C$:

1. How many degrees in arc $C\,M$? In $B\,N$? § 644.

2. In $B\,C + M\,N$? In $M\,N$? In $\angle D$? § 28.

*Therefore*—

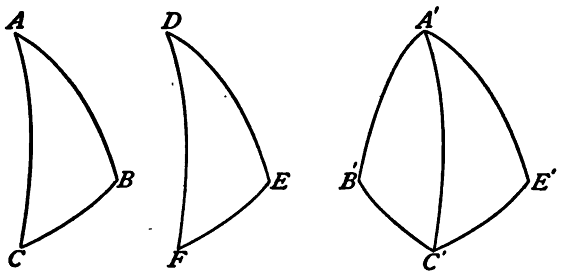
---

Ex. 385. What is the locus of a point in space the sum of the squares of whose distances from two fixed points is equal to the square of the distance between the two points?

Ex. 386. If two spheres are tangent either externally or internally, prove that the line joining their centers passes through the point of tangency. § 230.

## PROPOSITION XIX.

**669. Theorem.** *Two triangles on the same sphere, or equal spheres, having two sides and the included angle of one equal, respectively, to two sides and the included angle of the other are either equal or symmetrical.*




**Let A B C and D E F be two spherical △s in which ∠ A = ∠ D, A C = D F and A B = D E.**

CASE I. *When the parts are arranged in the same order.*

*To prove that triangle A B C is equal to triangle D E F in all respects.*

*Suggestion.* For method, see § 82 or § 664.

CASE II. *When the parts are arranged in reverse order.*

*To prove A′ B′ C′ and D E F are symmetrical.*

*Suggestion 1.* What are symmetrical △s? Construct △ *A′ E′ C′* symmetrical to *A′ B′ C′*.

2. Compare △s *A′ E′ C′* and *D E F*. (Case I.)

3. Compare *A′ B′ C′* and *D E F*. § 663.

*Therefore—*

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Ex. 387. The angles opposite the equal sides of an isosceles spherical triangle are equal. § 85.

## PROPOSITION XX.

**670. Theorem.** *Two triangles on the same sphere, or equal spheres, having two angles and the included side of one equal, respectively, to two angles and the included side of the other, are either equal or symmetrical.*

*Suggestion.* See figure and suggestions for § 669.

*Therefore—*

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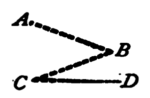
Ex. 388. One, and only one, surface of a sphere can be described through any four points not in the same plane.




*Suggestion 1.* Let $A$, $B$, $C$, and $D$ be the four given points.

*2.* What is the locus of points equally distant from $A$ and $B$?

*3.* What is the locus of points equally distant from $B$ and $C$?

*4.* What is the intersection of these two loci?

*5.* What is the locus of points equally distant from $C$ and $D$?

*6.* Does this last locus intersect the one referred to in Sug. 4? Why?

Ex. 389. A sphere can be inscribed in any tetrahedron. (§ 458.) Can more than one be inscribed? § 459.

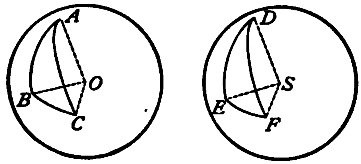
Ex. 390. Any side of a spherical triangle is less than the sum of the other two.

*Suggestion.* See § 656 (*c*).

Ex. 391. What is the locus of a point at a given distance from a given point, and also equidistant from two given points? Discuss the possibilities of the problem.

## Proposition XXI.

**671. Theorem.** *Two triangles on the same sphere, or equal spheres, having the three sides of one respectively equal to the three sides of the other, are either equal or symmetrical.*




**Let A B C and D E F be two spherical triangles on the same sphere, or equal spheres, having the three sides of one equal respectively to the three sides of the other.**

*To prove that A B C and D E F are either equal or symmetrical.*

*Suggestion 1.* Connect the vertices of each △ with the center of the sphere on which the △ is situated.

*2.* Compare the face ∠s of the trihedral ∠s *O* and *S*.

*3.* Compare the dihedral ∠s of the two trihedral ∠s. § 475.

*4.* Compare the spherical ∠s of one △ with the corresponding spherical ∠s of the other △. § 654.

*5.* Compare the △ *A B C* with the △ *D E F*, first, when the parts are arranged in the same order, and, second, when they are arranged in reverse order. § 669.

*Therefore—*

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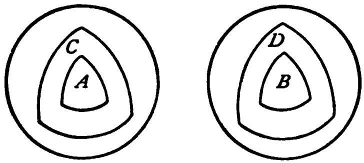
Ex. 392. Find point *x* that is *m* distant from one point, *n* distant from another and *o* distant from another? When is there one point *x*? When two? Is any other solution possible?

## Proposition XXII.

**672. Theorem.** *Two triangles on the same sphere, or equal spheres, having the three angles of one respectively equal to the three angles of the other, are either equal or symmetrical.*




**Let A and B represent two spherical triangles on the same sphere, or equal spheres, having the three angles of one equal, respectively, to the three angles of the other.**

*To prove that A and B are either equal or symmetrical.*

*Suggestion 1.* Let $C$ and $D$ be the polar $\Delta$s of $A$ and $B$, respectively.

2. Compare the sides of $C$ and $D$ (§ 668); the $\angle$s of $C$ and $D$; the sides of $A$ and $B$; the $\Delta$s $A$ and $B$.

*Therefore—*

---

Ex. 393. If two angles of a spherical triangle are equal, the triangle is isosceles.

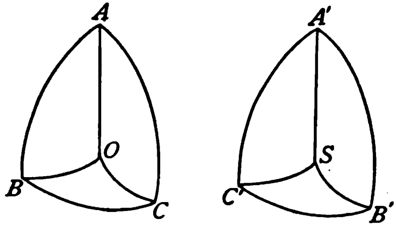
*Suggestion.* Construct the polar of the given $\Delta$.

Ex. 394. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base is perpendicular to the base and bisects the vertical angle.

Ex. 395. Determine a point, $x$, at a given distance from a given point, equidistant from two parallel planes and equidistant from two given points.

## Proposition XXIII.

**673. Theorem.** *Two symmetrical spherical triangles are equal in area.*




**Let A B C and A′ B′ C′ represent two symmetrical spherical triangles; i. e., let AB=A′ B′, AC = A′ C′, BC = B′ C′, ∠ A=∠ A′, ∠ B=∠ B′ and ∠C=∠ C′.**

*To prove that the triangles A B C and A′ B′ C′ are equal in area.*

*Suggestion 1.* Let *O* be the pole of a small ⊙ through the points *A*, *B*, and *C*. Draw the arcs of great ⊙s *O A*, *O B*, and *O C*.

*2.* Compare the arcs *O A*, *O B*, and *O C*. § 642.

*3.* At *A′*, draw *A′ S*, the arc of a great circle making ∠ *B′ A′ S* = ∠ *B A O*, and take *A′ S* = *A O*. Draw the arcs of great ⊙s *S C′* and *S B′*.

*4.* Compare △s *A O B* and *A′ S B′*; ∴ arcs *S A′* and *S B′* (Sug. 2); ∴ △s *A O B* and *A′ S B′* in respect to area. § 664.

*5.* Compare ∠s *O B C* and *S B′ C′*; △s *O B C* and *S B′ C′*; ∴ arcs *S C′* and *S B′*; ∴ △s *O B C* and *S B′ C′* in respect to area.

*6.* Compare △s *O A C* and *S A′ C′* in respect to area.

*7.* Compare △s *A B C* and *A′ B′ C′* in respect to area.

*Therefore—*

If $O$, the pole, is without the spherical triangle, one of the three triangles must be subtracted from the sum of the other two to equal $A\ B\ C$. Make a drawing to illustrate.

The pole may fall upon one of the arcs, in which case the above construction will produce but two triangles, whose sum is equal to $A\ B\ C$.

**674.** A **lune** is a portion of the surface of a sphere included between two semi-circumferences of great circles, as $A\ B\ C\ D$.

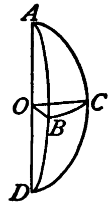



FIG. 11.

**675.** The **angle of a lune** is the angle of the two semi-circumferences which bound the lune, as the angle of the semi-circumference, $B\ A\ C$, is the angle of the lune $B\ A\ C\ D$.

**676.** COROLLARY I. *The angle of a lune is equal to the dihedral angle formed by the intersection of the semi-circumferences bounding the lune.*

**677.** COROLLARY II. *The angle of a lune is measured by the arc of a great circle described from either vertex of the lune as a pole and intercepted by the semi-circumferences.*

**678.** The **edge of a lune** is the edge of the dihedral angle formed by the intersection of two semi-circles whose semi-circumferences bound the lune.

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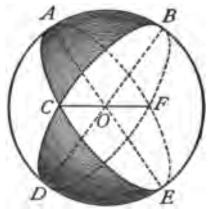
Ex. 396. What is the locus of a point in space which is at a given distance, $a$, from a given plane, and at a given distance greater than $a$ from a given point in the given plane?

Ex. 397. What is the locus of a point at a given distance from a given straight line, indefinite in length?

## Proposition XXIV.

**679. Theorem.** *If two arcs of great circles intersect on the surface of a hemisphere, the sum of the areas of the opposite spherical triangles thus formed is equal to the area of a lune whose angle is the angle of the intersecting arcs.*




**Let the arcs of great circles A C E and B C D intersect at the point C on the hemisphere A B E D, thus forming the opposite spherical triangles A C B and D C E.**

*To prove that the sum of the areas of the triangles A C B and D C E is equal to the area of a lune whose angle is D C E.*

*Suggestion 1.* Extend the arcs *A C E* and *B C D* to complete the great ⊙s *A C E F* and *B C D F*.

*2. C D F E* is a lune whose ∠ is *D C E*. Why?

*3.* To compare the △s *A C B* and *D E F*: Compare arcs *A C* and *E F*; *C B* and *D F*; *A B* and *D E*. How do the △s *A C B* and *D E F* compare in respect to area? Give auth.

*4.* Compare △s *A C B* and *C D E* with lune *C D F E*.

*Therefore—*

Query: Are △s *A B C* and *D E F* symmetrical, or are the parts arranged in the same order?

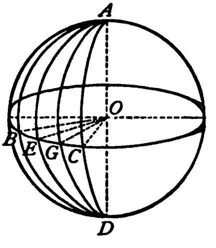
## Spherical Areas.

**680.** A **spherical degree,** or a degree of spherical surface, is one three hundred sixtieth of the surface of a hemisphere.

**681.** COROLLARY. *A spherical degree is one seven hundred twentieth of the surface of a sphere.*

## PROPOSITION XXV.

**682. Theorem.** *The area of a lune is to the area of the surface of a sphere as the angle of the lune is to four right angles*




**Let A B C D represent a lune whose angle is B A C.**

*To prove that the area of A B D C is to the area of the surface of the sphere as the angle B A C, or its measure B O C, is to four right angles.*

Let $B\ C$ be an arc of a great ⊙ whose poles are $A$ and $D$, and let $O$ be the center of the sphere.

CASE I. *When* $\angle\ B\ O\ C$ *and four right* $\angle$*s are commensurable.*

Employ the method of Case I in §§ 215, 290 and 447.

CASE II. *When* $\angle\ B\ O\ C$ *and four right* $\angle$*s are incommensurable.*

Employ the method of Case II in the same sections.

683. COROLLARY. *A lune contains twice as many spherical degrees as its angle contains angular degrees.*

Let $S$ represent the number of spherical degrees in the surface of the lune, and $A$ the number of angular degrees in the ∠ of the lune. Then, as there are 720 spherical degrees in the surface of a sphere, it is evident from the proposition that $\frac{S}{720} = \frac{A}{360}$. Hence, $S = 2A$.

684. If three great circles are perpendicular to one another, eight spherical triangles are formed, each having three right angles. Each of these triangles is a **tri-rectangular triangle.**

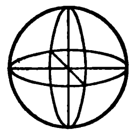


FIG. 12.

685. COROLLARY. *A tri-rectangular triangle has ninety spherical degrees.*

## PROPOSITION XXVI.

686. **Theorem.** *The sum of the sides of a convex spherical polygon is less than the circumference of a great circle of the sphere.*

*Suggestion 1.* Connect each vertex of the polygon with the center of the sphere, thus forming a polyhedral ∠.

*2.* Compare the sum of the face ∠s of the polyhedral ∠ with four right ∠s.

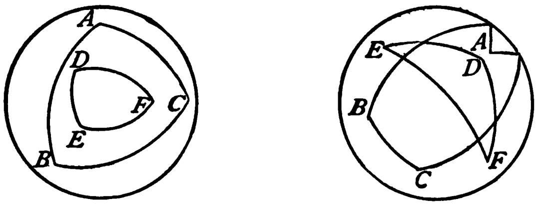
*3.* Compare the sum of the sides of the spherical polygon with the circumference of a great ⊙.

*Therefore—*

687. SCHOLIUM. The sides of a spherical polygon are usually expressed in degrees. Hence, the theorem may be stated thus: The sum of the sides of a convex spherical polygon is less than 360°, or four right angles.

## Proposition XXVII.

**688. Theorem.** *The sum of the angles of a spherical triangle is greater than two and less than six right angles.*




**Let A B C represent a spherical triangle.**

*To prove that the sum of the angles A, B and C is greater than two and less than six right angles.*

*Suggestion 1.* Let $D\ E\ F$ represent the polar $\triangle$ of $A\ B\ C$. Then $\angle A = 180° - E\ F$. Why?

2. Find $\angle$s $B$ and $C$ in terms of $D\ F$ and $D\ E$, respectively.

3. Find the sum of the $\angle$s $A$, $B$ and $C$ and express the result in its simplest form.

4. Since $E\ F + D\ F + D\ E$ is less than $360°$ (§ 686), what may be said of the sum of the $\angle$s $A$, $B$ and $C$?

*Therefore—*

**689.** Scholium. A spherical triangle, unlike a plain triangle, may have two or three right angles, or two or three obtuse angles.

Corollary. *If a spherical triangle varies in size, the sum of its angles approaches six right angles as a major and two right angles as a minor limit; and the sum of its sides approaches the circumference of a great circle as a major and zero as a minor limit.*

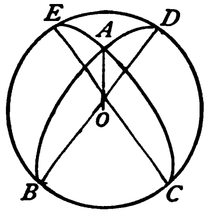
**690.** The **spherical excess** of a spherical triangle is the excess of the sum of its angles over two right angles.

If, in $\triangle ABC$, $A = 110°$, $B = 87°$ and $C = 140°$, the spherical excess $= 157°$.

**691.** COROLLARY. *If a spherical triangle varies in size, its spherical excess varies, and approaches four right angles as its major and zero as its minor limit.* § 690.

## PROPOSITION XXVIII.

**692. Theorem.** *The number of spherical degrees in a spherical triangle is equal to the number of angular degrees in its spherical excess.*




**Let A B C be a spherical triangle.**

*To prove that the number of spherical degrees in A B C is equal to the number of angular degrees in* $\angle A + \angle B + \angle C - 180°$.

*Suggestion 1.* Let one side of the $\triangle$, as $BC$, be extended to form a complete great ⊙, $BCDE$. Let $BA$ and $CA$ be extended to meet the great ⊙, $BCDE$ in $D$ and $E$, respectively.

*2.* $BCD$, $BAD$ and $BED$ are semi-circumferences of great ⊙s. Why?

*3.* $\triangle ABC + \triangle AED =$ lune whose $\angle$ is $A$. Why?

*4.* $\triangle ABC + \triangle ACD =$ lune whose $\angle$ is $B$. Why?

*5.* $\triangle ABC + \triangle ABE =$ lune whose $\angle$ is $C$. Why?

*6.* How many spherical degrees in a lune whose $\angle$ is $A$? In a lune whose $\angle$ is $B$? In a lune whose $\angle$ is $C$? Give auth. § 683.

*7.* How many spherical degrees in a hemisphere?

*8.* If the number of spherical degrees in $\triangle ABC$ be represented by $m$, then, from the equations in Suggestions 3, 4 and 5, $2m + 360 = 2A + 2B + 2C$.

$\therefore\ m = \angle A + \angle B + \angle C - 180°$.

*Therefore*—

**693.** SCHOLIUM. To say that a spherical triangle contains a certain number, $n$, of spherical degrees is simply to say that the surface of the triangle is equal to $\frac{n}{360}$ of the surface of a hemisphere, or to $\frac{n}{720}$ of the surface of a sphere. When the area of the surface of the sphere has been determined the area of the spherical triangle can be determined. Exs. 401, 402 and 403.

**694.** A **zone** is that portion of the surface of a sphere which is included between two parallel planes, as the surface $M$.

(*a*) The circumferences of the sections of the sphere made by the planes are the **bases of the zone,** as 1 2 and 3 4.

(*b*) A line perpendicular to the planes and intercepted between them is the **altitude of the zone.**

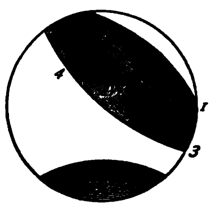


FIG. 13.

**695.** A **spherical segment** is a portion of a sphere included between two parallel planes, as 2 3.

(*a*) The sections of the sphere made by the planes are the **bases** of the segment, and the line perpendicular to the planes and intercepted between them is the **altitude** of the segment.

The spherical surface of a segment is a zone. Draw illustration from geography.

The portion of a sphere cut off by any plane is a spherical segment, for it is included between the cutting plane and a plane tangent to the sphere and parallel to the cutting plane, as *N*, Fig. 13. For the same reason, the portion of the surface of a sphere cut off by any plane is a zone, *N*, Fig. 13. In this case the segment and zone have but one base. Find illustration in geography.

**696.** If a semi-circle be revolved about its diameter as an axis, a sphere is generated; any arc of the semi-circumference generates a zone.

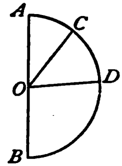



FIG. 14.

**697.** A **spherical sector** is that portion of a sphere generated by the revolution of a circular sector about a diameter.

The pupil should form a mental picture of the different varieties of spherical sectors and describe them. For example, if the semi-circle *A D B*, Fig. 14, is revolved about the diameter, the circular sector, *A O C*, generates a spherical sector whose surface is a zone of one base generated by the arc *A C*, and a convex conical surface generated by the radius *O C*. The circular sector *C O D* generates a spherical sector bounded by a zone of two bases, a convex conical surface generated by *O D*, and a concave conical surface generated by *O C*.

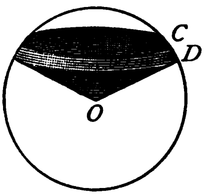



FIG. 15.

Generate many spherical sectors and describe them.
Make many drawings to illustrate, as Fig. 15.

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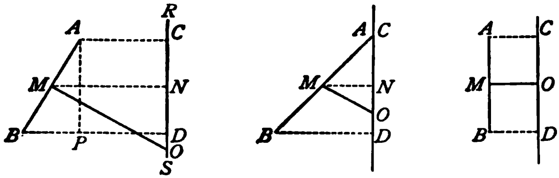
Ex. 398. Construct a semi-circle, and in it a circular sector, which, if revolved, generates a spherical sector having two concave conical surfaces. Describe the zone.

Ex. 399. Construct a circular sector which generates a spherical sector whose surface is a concave conical surface, a plane surface and a zone.

Ex. 400. Is a hemisphere a spherical sector? Why?

## Proposition XXIX.

**698. Theorem.** *The area of the surface generated by the revolution of a straight line about an axis in its plane, but not intersecting it, is equal to the product of the projection of the line upon the axis by the circumference of a circle whose radius is a perpendicular erected at the middle point of the line and limited by the axis.*




**Let A B be the straight line which revolves about R S as an axis in the plane of A B R S, but not intersecting R S; C D the projection of A B upon R S, and M O the perpendicular erected at the middle point of A B and limited by R S.**

*To prove that the area of the surface generated by A B is equal to the product of C D by the circumference of a circle whose radius is M O; i. e.,* $2\pi\, M O \times C D$.

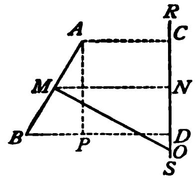
Case I. *When A B is not parallel to, and does not meet, the axis.*

*Suggestion 1.* Draw $M N \perp$ to $R S$. Connect $A$ and $C$, $B$ and $D$. Draw $A P \parallel$ to $R S$. $A C$ and $B D$ are $\perp$ to $R S$. Why?

*2.* $A B$ generates the convex surface of the frustum of a cone of revolution. What is the area of this surface?

*3.* The lateral area of the frustum, by § 610, $= 2\pi\, M N \times A B$. Hence, equate $2\pi\, M N \times A B$, with the hypothesis $2\pi\, M O \times C D$, and test whether your statement is true.

*4.* If the two products are equal, $\frac{AB}{MO} = \frac{CD}{MN}$. § 263. Test whether $\frac{AB}{MO} = \frac{CD}{MN}$ by comparison of $\Delta$s $ABP$ and $OMN$. $CD = AP$. Why?

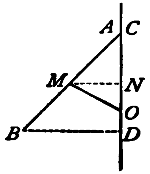



*5.* If the proportion holds true in the $\Delta$s named, what is your conclusion concerning the hypothetical equation in Sug. 3?

*Therefore—*

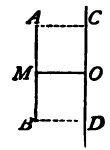
CASE II. *When $AB$ meets the axis.*

*Suggestion.* This case is obtained from Case I, by making $AC$ equal to zero. Examine the reasoning in Case I, and see that it holds when $AC$ = zero. The lateral area of the cone = $AB \times 2\pi MN$. Why?




CASE III. *When $AB$ is parallel to the axis.*

*Suggestion.* This case is obtained from Case I, by making $AC = BD$. Examine the reasoning in Case I, and see that it holds when $AC = BD$.




*Therefore—*

QUERY: What is the name of the surface generated by $AB$ when $A$ is on the axis? When $AB$ is parallel to the axis?

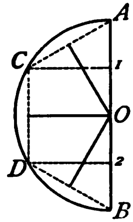
---

Ex. 401. How many spherical degrees are there in a spherical triangle whose angles are 200°, 140° and 100°?

Ex. 402. What part of the surface of a sphere is a spherical triangle whose angles are 120°, 140°, 160°?

PROPOSITION XXX.

**699. Theorem.** *The area of a sphere is equal to the product of its diameter by the circumference of a great circle.*




**Let A C D B represent a semi-circle; A B its diameter and O the center of A B. Let A C D B be revolved about A B to produce a sphere.**

*To prove that area of the surface of the sphere is equal to the product of A B by the circumference of great circle,* $2\pi R A B$.

*Suggestion 1.* Divide *A C D B* into any number of equal arcs, draw the chords of the arcs, as *A C*, etc., erect ⊥s at the middle of the respective chords, and extend to *A B*. Where do they meet *A B*? Why? Compare them in length.

*2.* What is the area of the surface generated by *A C*, by *C D*, etc.? What by the sum of all the chords? Express algebraically, and reduce to its simplest form.

*3.* Let the number of chords be indefinitely increased. The surface generated is a variable. What is its limit?

*4.* The apothem is a variable. What is its limit? Why is $2\pi$ apothem × *A B* a variable? What is its limit? Why? 214 (*a*).

*5.* The equation in Sug. 2 is always true as the variables approach their limits. Compare their limits. Give auth.

*Therefore—*

**700.** COROLLARY I. *The truth of the theorem may be expressed by the formula.*

$$S = 2\pi R D, \text{ or } 4\pi R^2$$

*in which S represents the surface of the sphere, R its radius and D its diameter.*

**701.** COROLLARY II. *The area of the surface of a sphere is equal to the area of four of its great circles.* § 700.

**702.** COROLLARY III. *The area of the surface of a sphere is equal to the area of a circle whose radius is the diameter of the sphere, or* $\pi D^2$.

**703.** COROLLARY IV. *The areas of the surfaces of two spheres have the same ratio as the squares of their radii or the squares of their diameters.*

**704.** COROLLARY V. *The area of a spherical degree is equal to* $\frac{4\pi R^2}{720}$. § 693.

**705.** COROLLARY VI. *The area of a zone is equal to the altitude of the zone by the circumference of a great circle.*

Use the method of demonstrating the proposition. The area can be expressed by the formula $2\pi R H$, in which $R$ represents the radius of the sphere, and $H$ the altitude of the zone.

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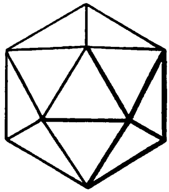
Ex. 403. What part of the surface of a sphere is a spherical triangle, each angle of which is 90°? How many spherical degrees in the same triangle?

Ex. 404. The sides of a triangle, on a sphere whose radius is 10 feet, are respectively 95°, 117° and 94°. Find the area in square feet of its polar triangle.

Ex. 405. Assuming the earth to be a sphere whose diameter is 7,912 miles, how many square miles upon its surface?

## Proposition XXXI.

**706. Theorem.** *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*



*Suggestion 1.* Circumscribe a polyhedron about the sphere.

*2.* Join each vertex of the polyhedron with the center of the sphere, and pass planes through these lines and the edges of the polyhedron. Pyramids are thus formed. Why?

*3.* Compare the altitude of these pyramids with the radius of the sphere.

*4.* What is the volume of each pyramid?

*5.* What is the volume of the polyhedron? Reduce algebraically.

*6.* Circumscribe a polyhedron of a greater number of sides about the sphere. What is its volume?

*7.* Finish the demonstration.

*Therefore —*

Another method: Let a regular polyhedron be inscribed in a sphere. Drop ⊥s from the center of the sphere to the faces of the polyhedron and prove the proposition.

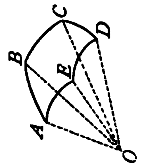
**707.** Corollary I. *The volume of a sphere can be expressed by the formula* $V = \frac{4}{3}\pi R^3$ (§ 700), *in which V represents the volume and R the radius of the sphere.*

**708.** Corollary II. *The volumes of two spheres have the same ratio as the cubes of their radii, or cubes of their diameters.*

**709.** Corollary III. *The volume of a spherical sector is equal to the area of its zone by one-third the radius of the sphere, and may be expressed by the formula* $\frac{2}{3}\pi R^2 H$.

See method of demonstration, § 706.

**710.** A **spherical pyramid** is a solid bounded by a spherical polygon and the planes of the sides of the polygon, as *O-A B C*, etc.




(*a*) The **vertex** of the pyramid is at the center of the sphere.

(*b*) The spherical polygon is the base of the spherical pyramid.

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Ex. 406. The volume of a spherical pyramid is equal to the area of its base multiplied by one-third of the radius of the sphere.

*Suggestion.* Employ the method in § 706.

Ex. 407. Prove that the volume of a sphere is twice the volume of a cone whose altitude is equal to the diameter of the sphere, and the radius of whose base is equal to the radius of the sphere.

Ex. 408. Assuming the diameter of the earth to be 8,000 miles, and that of the moon 2,000; how do the amounts of light reflected from them to a point in space equally distant from each compare?

Ex. 409. Find, in square feet, the area of a spherical triangle whose angles are 95°, 149°, and 216°, the radius of the sphere being 15 inches.

Ex. 410. On the same sphere, or equal spheres, zones of equal altitudes are equal in area.

Ex. 411. Find the volume of a spherical sector having a zone with an altitude of 10 inches on a sphere with a radius of 20 inches.

Ex. 412. Find the volume and area of the surface of the sphere in exercise 411.

Ex. 413. A sphere is cut by parallel planes so that the diameter is divided into ten equal parts. Compare the areas of the zones; also the volumes of the spherical sectors whose spherical surfaces are the respective zones.

Ex. 414. If the average specific gravity of the earth is 5.6, what is its weight expressed in tons.

Ex. 415. Find the angles of an equiangular spherical triangle whose surface is one-twelfth of the surface of a sphere.

Ex. 416. The radius of a sphere is 3 inches, and the area of a spherical triangle $ABC$ on this sphere is 18.7″. The angles $A$ and $B$ are 72° and 115°, respectively. Find the angle $C$ of the spherical triangle.

Ex. 417. The dimensions of a rectangular parallelopiped are 3, 4, and 12 feet. Find the circumference of a great circle of the circumscribing sphere.

Ex. 418. With the same assumption as that of exercise 408, what is the ratio of the volumes of the earth and moon?

Ex. 419. A triangle on a 12-inch globe has for its angles 140°, 119°, and 196° respectively; compute its area.

Ex. 420. Prove Proposition, § 673, when the pole of the small circle through $A$, $B$ and $C$ is without the triangle.

Ex. 421. If the area of the convex surface of a right circular cone is twice the area of its base, prove that the slant height of the cone is equal to the diameter of its base.

Ex. 422. The radius of the base of a right circular cone is 5 inches, and the number of square inches in the area of the convex surface of the cone is equal to the number of cubic inches in the volume of the cone. Find the altitude and the slant height of the cone.

Ex. 423. A pyramid whose altitude is $\sqrt[3]{16}$ feet is cut into two parts of equal volume by a plane parallel to the base. Find the distance of the cutting plane from the vertex.

Ex. 424. Which generates the greater volume, a rectangle revolved about its longer or shorter side?

Ex. 425. Demonstrate the proposition, § 706, by use of the varying solid whose surface is the varying surface in § 699.

Ex. 426. A plane was passed parallel to the base of a cone cutting the altitude into two equal parts. Compare the two parts into which the cone was divided.

Ex. 427. A farmer's "circular" watering tank is $2\frac{1}{2}'$ deep and has a diameter of $10\frac{1}{2}'$. How many barrels does it hold? (Indicate operation and abbreviate by cancellation).

Ex. 428. Find the volume of a cube $4'$ on an edge by Sug. 1-5, § 706.

Ex. 429. What is the locus of the center of a sphere having a given radius?

*1.* Whose surface passes through a given point.

*2.* Which is tangent to a given plane?

*3.* Which is tangent to a given line?

*4.* Which is tangent to a given sphere?

Ex. 430. Find the center of a sphere having a given radius:

*1.* Whose surface passes through three given points.

*2.* Which is tangent to a given plane and whose surface passes through two given points.

*3.* Which is tangent to two given planes and whose surface passes through a given point.

*4.* Which is tangent to three given planes.

*5.* Which is tangent to a given sphere, to a given plane, and whose surface passes through a given point.

*6.* Which is tangent to a given line, a given plane, and a given sphere.

Make other exercises in this group.

Ex. 431. The diameter of a sphere is equal to the altitude of a cone of revolution and of a cylinder of revolution, and the radius of the sphere is equal to the radius of the cone and of the cylinder. Prove that the volumes of the cone, sphere, and cylinder are proportional to the numbers 1, 2, and 3.

Ex. 432. An orange, $3\frac{1}{2}$ inches in diameter, sells at 50 cents a dozen, one 3 inches in diameter sells at 40 cents. Which is the better one to buy if they are of the same quality and the peeling on each is $\frac{1}{8}$ of an inch thick. If the larger orange is worth 50 cents, what is the smaller one worth?

## Review.

**711.** State formulæ for:

1. The volume of a cone of revolution.
2. The volume of a cylinder of revolution.
3. The volume of a frustum of a cone of revolution.

4. The lateral area of each figure of 1, 2, and 3.
5. The area of a sphere.
6. The volume of a sphere.
7. The area of a zone of a sphere.
8. The volume of a spherical sector.

State how to find:

1. The area of a spherical triangle.
2. The area of a spherical polygon of *n* sides.

## SUPPLEMENTARY PROPOSITIONS.

Ex. 433. (§ 96.)* The angle formed by the bisectors of the angles at the base of an isosceles triangle is equal to an exterior angle at the base of the triangle.

Ex. 434. A line drawn from one end of the base of an isosceles triangle perpendicular to the opposite side makes with the base an angle equal to one-half the vertical angle.

Ex. 435. (§ 126.) $ACB$ and $ADB$ are two triangles on the same side of $AB$, such that $AC$ is equal to $BD$, and $AD$ is equal to $BC$, and $AD$ and $BC$ intersect at $O$. Prove that $AOB$ is an isosceles triangle.

Ex. 436. If the vertical angle of an isosceles triangle is one-half as great as an angle at the base, the bisector of a base angle divides the given triangle into two isosceles triangles.

---

### PROPOSITION I. (§ 214.)*

**712. Theorem.** *If a variable approaches zero as a limit, any product of the variable by a finite number approaches zero as its limit.*

**Let V denote the variable which approaches zero as its limit, and let M denote any finite number.**

*To prove that $MV$ approaches zero as a limit.*

If $MV$ cannot have zero as a limit, let $S$ be any

* The references indicate where the Supplementary Propositions and Exercises may be inserted. They may be used anywhere after the reference but usually not before it.

assigned quantity, however small, by which amount it differs from zero. As $S$ is a finite magnitude, $\frac{S}{M}$ is a finite magnitude (Ax. 14, Cor.). Hence, $V$ as it approaches $O$ can be made smaller than $\frac{S}{M}$. (§ 211.) Hence, $M V$ can become smaller than $S$, but cannot equal zero. (Ax. 7, Cor.) Hence $M V$ can be made to differ from zero by less than any assigned quantity, as $S$. (Ax. 1.) Hence $M V$ approaches zero as a limit. § 212.

*Therefore*—

## Proposition II.

**713. Theorem.** *If a variable approaches zero as a limit, any quotient of the variable by a finite constant approaches zero as a limit.*

**Let V denote a variable that approaches zero as its limit, and let M denote any finite quantity.**

*To prove that* $\frac{V}{M}$ *approaches zero as a limit.*

$V$ has zero for its limit. Each value of $\frac{1}{M}$ of $V$ is smaller than $V$ (Ax. 14, Cor.), but cannot equal zero, hence must have zero for its limit. Ax. 9.

$\frac{V}{M} = \frac{1}{M}$ of $V$.

$\therefore \frac{V}{M}$ has zero for a limit.

*Therefore*—

## PROPOSITION III.

**714. Theorem.** *The variable which is produced by multiplying a given variable by a finite constant has for its limit the product of the limit of the variable by the constant.*

**Let V be a variable, L its limit, and M any finite quantity.**

*To prove $M\,V$ has $M\,L$ for a limit.*

$L - V$ has zero for its limit. § 212.

$M\,(L-V)$, or $M\,L-M\,V$, has zero for its limit. § 712.

$M\,V$ has $M\,L$ for its limit. § 211.

*Therefore—*

## PROPOSITION IV.

**715. Theorem.** *The variable which is produced by dividing a given variable by a finite constant has for its limit the quotient of the limit of the variable by the constant.*

**Let V be any variable, L its limit, and M any finite constant.**

*To prove that* $\frac{V}{M}$ *has* $\frac{L}{M}$ *for its limit.*

$L-V$ has zero for its limit. § 212.

$\frac{L-V}{M}$, or $\frac{L}{M}-\frac{V}{M}$, has zero for its limit. § 713.

$\therefore \frac{V}{M}$ has $\frac{L}{M}$ for its limit. § 211.

---

Ex. 437. In triangle $A\,B\,C$, angle $B$ is three times $A$, and $C$ is five times $A$; find each angle of the triangle.

### Proposition V.

**716. Theorem.** *If each of two variables has zero for its limit, their product has zero for its limit.*

**Let V and V′ be two variables, each having zero for its limit.**

*To prove $VV'$ has zero for its limit.*

$M$ $V$ has zero for its limit (§ 712) and cannot equal zero, and as $V'$ can become less than $M$, $VV'$ must approach zero as its limit. Why?

*Therefore—*

### Proposition VI.

**717. Theorem.** *If each of two or more variables has zero for its limit their sum has zero for its limit.*

**Let V, V′, V″, etc., represent variables having zero for their limits.**

*To prove that $V + V' + V''$, etc., has zero for its limit.*

**Let S be any assigned quantity to represent the difference between V + V′ + V″, etc., and its respective limit.**

$V$ and $V'$ can each become less than $\frac{S}{2}$ but cannot equal zero (Hyp.). Hence $V + V'$ can become less than $S$ and cannot equal zero.

In the same way each of $M$ variables can become less than $\frac{1}{M}$ of $S$. Hence, their sum can become less than $S$. As $S$ is any assigned quantity, however small, $V + V' + V'' +$ etc., has zero for its limit.

*Therefore—*

## PROPOSITION VII.

**718. Theorem.** *The sum of two or more variables, all increasing or all decreasing, has for its limit the sum of the limits of the variables.*

**Let V, V′, V″, etc., represent variables and L, L′, L″, etc., their respective limits.**

*To prove* $V + V' + V'' +$ *etc., has for its limit* $L + L' + L'' +$ *etc.*

$L - V$, $L' - V'$, $L'' - V''$, etc., each has zero for its limit. § 212.

Let $R$, $R'$, $R''$, etc., represent $L - V$, $L' - V'$, $L'' - V''$, etc., respectively.

Then $V = L - R$, $V' = L' - R'$, $V'' = L'' - R''$, etc. (§ 212), adding, $V + V' + V'' +$ etc., $= L + L' + L'' +$ etc., $- (R + R' + R'' +$ etc.). Ax. 2. (*c.*)

$L + L' + L'' +$ etc., $- V + V' + V'' +$ etc., $= R + R' + R'' +$ etc. Ax. 3.

$R + R' + R'' +$ etc., has zero for its limit. § 717.

$\therefore L + L' + L'' +$ etc. $- (V + V' + V'' +$ etc.), has zero for its limit.

$\therefore V + V' + V'' +$ etc. has $L + L' + L'' +$ etc., for its limit. § 211.

*Therefore—*

---

Ex. 438. If $A\,B$ and $A\,C$ are equal sides of a triangle, and if $B\,M$ and $C\,N$ are bisectors of angles $B$ and $C$ respectively, prove that triangles $A\,B\,M$ and $A\,C\,N$ are equal, and also that triangles $B\,C\,N$ and $C\,B\,M$ are equal.

### Proposition VIII.

**719. Theorem.** *The product of two variables both of which are either increasing or decreasing, has for its limit the product of the limits of the variables.*

**Let V and V′ represent the two variables, L and L′ their respective limits.**

*To prove that $V V'$ has $L L'$ for its limit.*

$L - V$ and $L - V'$ each has zero as its limit.

Let $R$ and $R'$ represent $L - V$, and $L - V'$ respectively.

$V = L - R$, $V' = L' - R'$.

$V V' = L L' - L' R - L R' + R R'$. Ax. 4.

$L L' - V V' = L' R + L R' - R R'$. Ax. 3.

$L' R$, $L R'$ and $R R'$ each has zero for its limit. Why?

$L' R + L R' + (-R R')$ has zero for its limit. Why?

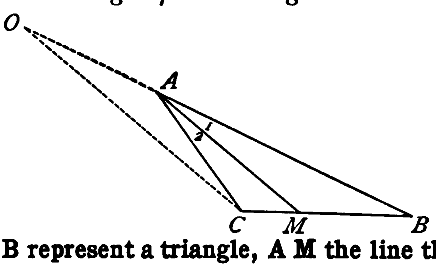
$L L' - V V'$ has zero for its limit. Why?

$V V'$ has $L L'$ for its limit. Why?

*Therefore—*

### Proposition IX. (§ 290.)

**720. Theorem.** *A line that bisects the vertical angle of a triangle divides the base into segments proportional to the two legs of the triangle.*




**Let A C B represent a triangle, A M the line that bisects**

**the vertical angle of the triangle and C M and M B the segments of the base.**

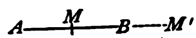
*To prove* $\frac{C\,M}{M\,B} = \frac{C\,A}{A\,B}$

*Suggestion 1.* Draw a line through $C \parallel$ to $A\,M$ and extend $A\,B$ to meet it at $O$. Compare $A\,O$ and $A\,C$.

2. Compare $\frac{C\,M}{M\,B}$ with $\frac{O\,A}{A\,B}$; $\frac{C\,M}{M\,B}$ with $\frac{C\,A}{A\,B}$.

*Therefore*—

**720 (a).** When a point, as $M$, is taken in a line between its extremities, the line is **divided internally.** $A\,B = A\,M + M\,B$. When a point, as $M'$, is taken in a line extended, the line is **divided externally** at the point. $A\,B = A\,M' + M'\,B$.




Verify the two equations. (Ax. 10.)

*Suggestion.* Let distance in the direction from $A$ to $B$ be positive, then distance in the direction from $B$ to $A$ is negative.

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Ex. 439. Converse of Proposition IX. A line drawn through the vertex of a triangle that divides the opposite side into segments proportional to the other two sides, bisects the angle.

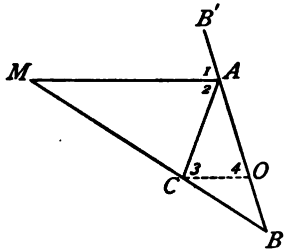
Use the figure in § 720 and prove the proposition.

Ex. 440. If $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A+B}{A-B} = \frac{C+D}{C-D}$.

Ex. 441. If the radius of one circle is the diameter of another, the circles are tangent to each other, and any line drawn from the point of contact to the outer circumference is bisected by the inner one.

## PROPOSITION X.

**721. Theorem.** *If an exterior angle of a triangle is bisected, the side opposite is divided externally into segments proportional to the other two sides.*




**Let A B C represent a triangle, B′ A C an exterior angle, and C M and M B the two segments into which the base is externally divided.**

*To prove that* $\frac{CM}{MB} = \frac{CA}{AB}$.

*Suggestion.* Draw a line through $C$ ∥ to the bisector, $AM$.

722. A line is divided in **mean and extreme ratio** if the ratio of the whole line to the greater segment is equal to the ratio of the greater segment to the lesser segment.

---

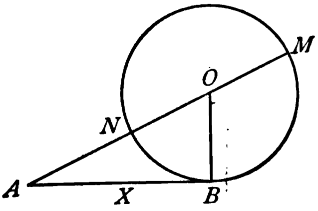
Ex. 442. State the converse of Proposition X.

Use figure § 721, and prove the proposition.

Ex. 443. If $ABC$ and $ABD$ are two triangles on the same base and on the same side of it, such that $AC$ equals $BD$, $AD$ equals $BC$ and $AD$ and $BC$ intersect at $O$, prove (1) that triangles $ABC$ and $ABD$ are equal in all respects; (2) that triangles $AOC$ and $BOD$ are equal in all respects; and (3) that triangle $AOB$ is isosceles.

## Proposition XI. (§ 314.)

**723. Problem.** *To divide a straight line in mean and extreme ratio.*




**Let A B be a given straight line.**

*To divide A B in mean and extreme ratio.*

*Suggestion 1.* At $B$ erect a $\perp$ to $A B$ equal to $\frac{1}{2}$ of $A B$, as $B O$.

With $O$ as a center, and $O B$ as a radius, describe a circumference. Draw a line through $A$ and $O$, cutting the circumference at $N$ and $M$.

On $A B$ lay off a distance $A X$ equal to $A N$.

2. $\frac{A M}{A B} = \frac{A B}{A N}$. Why? § 314. 4. $\frac{A B}{A N} = \frac{A N}{B X}$. Why?

3. $\frac{A N}{A B} = \frac{B X}{A N}$. Why? § 281. 5. $\frac{A B}{A X} = \frac{A X}{B X}$. Why?

*Therefore* —

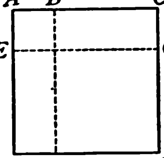
---

Ex. 444. A circle circumscribes an isosceles triangle, and tangents are drawn to the circle through the vertices of the triangle. Prove that these tangents form a second isosceles triangle, and that the two triangles cannot have equal vertical angles unless both are equilateral.

Ex. 445. In a given line determine a point which is equally distant from two given points not in the line.

### Proposition XII. (§ 330.)

**724. Theorem.** *The square described upon the sum of two lines is equal to the sum of the squares described upon the two lines plus twice the rectangle of the two lines.*




**Let AB and BC be two given lines, and AC their sum. Let AN be the square described upon AC.**

*To prove that A N equals the sum of the squares described upon A B and B C, plus twice the rectangle whose sides are A B and B C.*

*Suggestion 1.* Lay off *A E* equal to *A B*, and draw *E O* ∥ to *A C*; also draw *B M* ∥ to *C N*.

*2.* Study the parts of square *A N* and describe them in terms of the theorem.

**725.** Scholium. With the interpretation given in § 329 for the square of a line and the product of two lines, this proposition may be expressed thus:

If $AC = AB + BC$

then $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC$.

Compare this with the algebraic formula for the square of the sum of two numbers.

*Therefore*—

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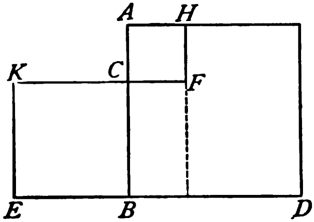
Ex. 446. Square 36, (30 + 6), by § 725 and compare your process, part by part, with the usual method.

### PROPOSITION XIII. (§ 330.)

726. **Theorem.** *The square described upon the difference of two lines equals the sum of the squares described upon the two lines minus twice the rectangle of the two lines.*




**Let A B and B C be two given lines, and A C their difference. Let A F be the square described upon A C.**

*To prove that A F equals the sum of the squares described upon A B and B C, minus twice the rectangle whose sides are A B and B C.*

*Suggestion 1.* Let $A D$ be the square described upon $A B$, and $K B$ the square described on $B C$.

2. Compare square $A F$ + twice $(A B \times C B)$ with $\overline{A B}^2 + \overline{B C}^2$.

State conclusion in form of theorem.

*Therefore —*

727. SCHOLIUM. With the interpretation given in § 329 for the square of a line and the product of two lines, this proposition may be expressed thus:

If $A C = A B - B C$, $\overline{A C}^2 = \overline{A B}^2 + \overline{B C}^2 - 2 A B \times B C$.

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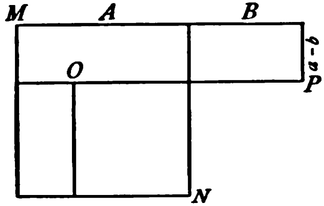
Ex. 447. Through two points an inch apart draw two parallel lines.

## Proposition XIV. (§ 330.)

**728. Theorem.** *The product of the sum of two lines and their difference is equal to the difference of their squares.*



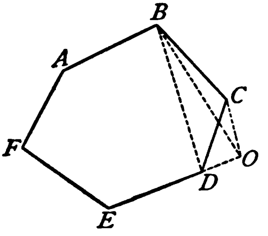

**Let A and B represent the two lines, M N the square upon A, O N the square upon B and M P the rectangle (a + b) by (a − b).**

*To prove that M N minus O N is equal to M P.*

Express algebraically. § 329.

## Proposition XV. (§ 336.)

**729. Problem.** *To construct a triangle equal in area to a given polygon.*




**Let A B C, etc., be a given polygon.**

*To construct a triangle equal in area to the polygon A B C, etc.*

*Suggestion 1.* Draw a diagonal, as $B D$, cutting off one vertex, as $C$.

*2.* Through $C$, the vertex cut off, draw a line ‖ to the diagonal $B D$.

*3.* Extend $E D$ until it meets the line ‖ to the diagonal, as at $O$. Connect $O$ with $B$.

*4.* Compare the areas of the $\triangle$s $C B D$ and $O B D$.

*5.* Compare the areas of the polygons $A B C$, etc., and $A B O E$, etc.

*6.* Compare the number of sides of the last polygon with that of the original polygon.

Continue the process until the polygon is reduced to a triangle.

*Therefore*—

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Ex. 448. The altitude of a trapezoid is 3 ft. and the bases are 8 and 12 ft. respectively. Extend the non-parallel sides until they meet, and find the areas of the two triangles of which the trapezoid is the difference.

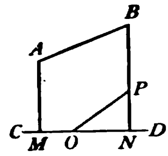
Ex. 449. (§ 341.) If the center of each of two equal circles lies on the circumference of the other, the square on the common chord is equal to three times the square on the radius.

Ex. 450. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

Ex. 451. What is the ratio of the areas of two similar triangles whose homologous sides have the ratio $\frac{3}{5}$.

Ex. 452. Construct a pentagon. Find the number of square inches in its area by § 729 and § 336 and compare results.

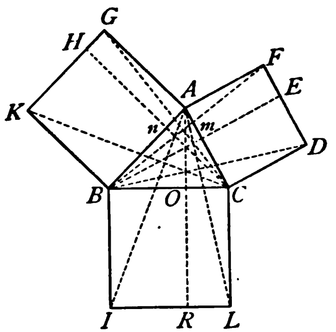
730 (a). The **projection of a point** upon a straight line is the foot of the perpendicular from the point to the line.




(b). The **projection of a straight line** upon another straight line is that part of the second line included between the projections of the extremities of the given line. The projection of $A$ upon $CD$ is $M$, of $B$ is $N$; of $AB$ upon $CD$ is $MN$, of $OP$ is $ON$.

## Proposition XVI. (§ 341.)

731. **Theorem.** *The square upon the side opposite an acute angle of a triangle equals the sum of the squares upon the other two sides minus twice the product of one of the two sides by the projection of the other side upon that side.*




**Let ABC be a triangle of which the angle A is acute, and let BL, AD, and AK be the squares described upon the sides BC, AC, and AB, respectively.**

*To prove that the square $BL$ equals the square $AD$*

*plus square A K minus twice A B times An the projection of A C on A B.*

*Suggestion 1.* From $A$, draw $A\,O \perp$ to $B\,C$, and extend it to $I\,L$. From $B$, draw $Bm \perp$ to $A\,C$, and extend it to $F\,D$. From $C$, draw $Cn \perp$ to $A\,B$ and extend it to $K\,G$.

2. Draw $A\,I$, $A\,L$, $B\,D$, $B\,F$, $C\,K$ and $C\,G$.

3. Compare rectangle $C\,E$ with rectangle $C\,R$, $H\,B$ with $B\,R$. See method § 341.

Square $B\,L$ with rectangles $H\,B + C\,E$.

4. Compare square $B\,L$ with squares $A\,K + A\,D$.

5. Compare rectangles $A\,H$ and $A\,E$.

6. $A\,H = A\,B \times$ the projection of $A\,C$ upon $A\,B$. Give auth.

Complete the demonstration.

Express the theorem algebraically.

*Therefore* —

---

Ex. 453. Construct a triangle equal in area to a given triangle, two sides of the required triangle being given. Show when there are two solutions, when one solution, and when no solution.

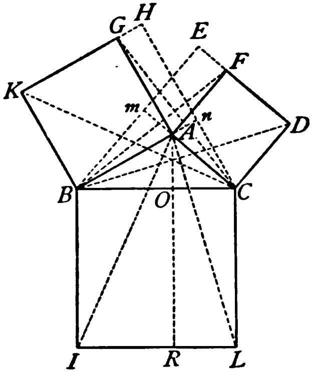
Ex. 454. If, from a point without a circle, two secants be drawn whose external segments are 8 inches and 7 inches, while the internal segment of the latter is 17 inches, what is the internal segment of the former?

Ex. 455. The sides of a triangle are 5, 6, and 7, and the side corresponding to 6, in a similar triangle, is 36; find the other two sides of the triangle.

Ex. 456. Two isosceles triangles have equal vertical angles. Prove that they are similar.

## PROPOSITION XVII. (§ 341.)

**732. Theorem.** *The square upon the side opposite an obtuse angle of a triangle is equal to the sum of the squares upon the other two sides plus twice the product of one of the sides by the projection of the other side upon that side.*




**Let A B C be a triangle, of which the angle A is obtuse; let B L, A K, and A D be squares upon the sides B C, A B, and A C, respectively, and let An be the projection of A C upon A B.**

*To prove that the square B L is equal to the sum of the squares A K and A D, plus twice the product of A B by An.*

*Suggestion 1.* From each vertex of the △ draw ⊥s to the opposite side of the opposite square, or the side extended, as *C H*, *A R*, and *B E*. Draw from the same points lines to the opposite vertices of the opposite squares, as *C K*, *C G*, *B F*, *B D*, *A I* and *A L*.

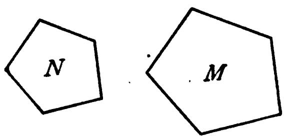
*2.* See method in § 731, and complete the demonstration.

Compare the demonstration of the propositions in sections 341, 731 and 732.

State one method of construction so that it may apply to each of the three propositions.

### PROPOSITION XVIII. (§ 338.)

**733. Problem.** *To find two straight lines which have the same ratio as two given similar polygons.*




**Let M and N be two given similar polygons.**

*To find two lines whose ratio is equal to the ratio* $\frac{M}{N}$.

*Suggestion 1.* What is the ratio of $M$ to $N$ in terms of their sides? § 338.

*2.* Where has the ratio of two squares been compared to two lines? Ex. 209.

*3.* Make the required construction.

---

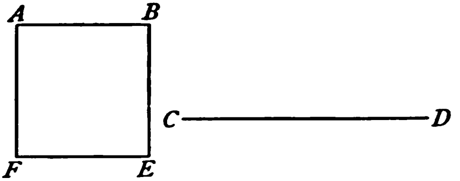
Ex. 457. If three similar polygons be constructed on the three sides of a right triangle, prove that the area of the polygon constructed on the hypotenuse equals the sum of the areas of the polygons constructed on the other two sides.

*Suggestion.* See §§ 341 and 338.

Ex. 458. All equal chords of any circle are tangents to some other circle.

PROPOSITION XIX. (§ 344.)

**734. Problem.** *To construct a rectangle in which the sum of the base and altitude is equal to a given line, and the area is equal to the area of a given square.*




**Let C X equal the altitude, X D the base of the rectangle and A E the given square.**

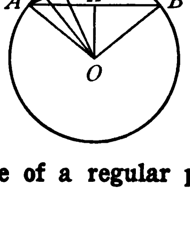
*To find X in line C D.*

*Suggestion 1.* $CX \times XD = AB^2$ or $AB \times BE$.

*2.* Make a proportion from above equation, and complete the solution. § 310, II.

PROPOSITION XX. (§ 369.)

**735. Theorem.** *If the number of sides of a regular inscribed polygon is increased indefinitely, the apothem is a variable which approaches the radius as a limit.*




**Let A B be a side of a regular polygon, and O H its apothem.**

*To prove that if the number of sides of the polygon is increased indefinitely, $O H$ is a variable which approaches $O A$ as a limit.*

*Suggestion 1.* What are the tests for the limit of a variable? § 211 (*a*).

*2.* In △ $A O H$ compare $O A$ and $O H$; $O A - O H$ with $A H$; $O A - O H'$ with $A H'$, etc.; $A H$ with $A B$, $A H'$ with $A B'$, etc.

*3.* By continually increasing the number of sides of the polygon, one side, as $A B$, may be made less than any assigned line, however short. *Ax.* 14, Cor. I.

*4.* $O A - O H$ is a variable. What is its limit? Why? Suggestions 2 and 3. What relation does $O A$ sustain to $O H$? (§ 212.) Apply the four tests for the limit of a variable. § 211 (*a*).

*Therefore* —

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Ex. 459. Find the side of a square equal in area to a rectangle whose sides are 6 and 9.

Ex. 460. Given two similar triangles, construct a third triangle similar to the other two whose area shall be equal to the sum of the areas of the other two.

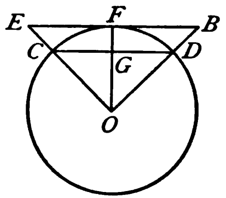
Ex. 461. Find the altitude of an equilateral triangle whose side is 10 inches. Find the side when the altitude is 10 inches.

Ex. 462. Construct a parallelogram having a given angle, whose base and altitude together are equal to a given line, and whose area is equal to the area of a given square.

Ex. 463. Find the locus of a point equally distant from the circumferences of two equal non-intersecting circles.

PROPOSITION XXI. (§ 370.)

**736. Theorem.** *If the number of sides of a regular inscribed polygon be increased indefinitely, the perimeter of the polygon is a variable which approaches the circumference of the circle as a limit.*




**Let C D represent the side of a regular inscribed polygon and p its perimeter. Let C be the circumference of the circumscribed circle whose center is O.**

*To prove that as he number of the sides of the polygon increases, $p$ is a variable which approaches the circumference of the circle as a limit.*

*Suggestion 1.* Let $EB$ be the side of a circumscribed polygon similar to the inscribed polygon, and let $P$ denote the perimeter of the circumscribed polygon. Let $r$ denote the radius and $a$ the apothem of the inscribed polygon.

2. $\frac{P}{p} = \frac{r}{a}$. Give auth. 3. $\frac{P-p}{P} = \frac{r-a}{r}$. Give auth.

4. $P-p = (r-a) \times \frac{P}{r}$. Give auth.

5. $\frac{P}{r}$ continually decreases, since $P$ decreases and $r$ remains unchanged; also $r-a$ diminishes indefinitely (§ 735). Hence $(r-a) \times \frac{P}{r}$ diminishes indefinitely, but

cannot be made absolutely zero. (§ 712.) Hence $P-p$ diminishes.

6. $c-p$ is less than $P-p$, but still cannot be made equal to zero, therefore, $c$ is the limit of $p$. § 212.

∴ The circumference of the ⊙ is the limit of the perimeter of the inscribed polygon when the number of sides of the po'ygon is made to increase indefinitely. *Q. E. D.*

Apply the four tests of the limit of a variable in drawing your conclusion.

**737.** Corollary. The circumference of a circle is the limit of the perimeter of the regular circumscribed polygon if the number of its sides be indefinitely increased. § 362.

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Ex. 464. In $A B$, the diameter of a circle, or, in $A B$ extended, take any point $C$, and draw $C D$ perpendicular to $A B$; if $A$ be joined with any point $P$, in $C D$, and $A P$ meet the circumference at $Q$; then $A P \times A Q = A C \times A B$. That is $A P \times A Q$ is a constant.

Show the varying positions of $A P$.

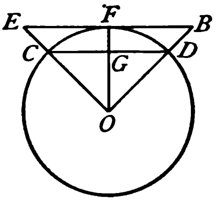
Ex. 465. If $A$ is a given point, and $P$ any point in a given straight line, and if a point $Q$ be taken in the line joining $A$ and $P$, so that $A P \times A Q$ is constant, then, as $P$ moves along the given line, $Q$ will move on the circumference of a circle which passes through $A$.

Ex. 466. If $A B$ be the diameter of a circle, and if a point $P$ be taken on any chord $A Q$, or $A Q$ extended so that $A P \times A Q$ is constant, the locus of $P$ is a straight line perpendicular to $A B$.

Ex. 466 (*a*). The sum of the squares of the diagonals of any quadrilateral equals twice the sum of the squares of its two diameters.

## PROPOSITION XXII. (§ 371.)

**738. Theorem.** *If the number of sides of a regular inscribed polygon is increased indefinitely, the area of the polygon is a variable which approaches the area of the circle as a limit.*




**Let C D represent a side, A the area of an inscribed polygon and M the area of the circle.**

*To prove that if the number of sides of the polygon is indefinitely increased, A is a variable whose limit is M.*

*Suggestion 1.* Let $r$ denote the radius and $a$ the apothem of the inscribed polygon, also, let $p$ denote the perimeter of the inscribed polygon, $P$ the perimeter of a similar circumscribed polygon, and $A^1$ its area. Let $EB$ be a side of the circumscribed polygon homologous to the side $CD$ of the inscribed polygon.

2. Trapezoid $EBDC = (EB + CD) \times \frac{1}{2}(r - a)$. Why?

3. $A^1 - A = (P + p) \times \frac{1}{2}(r - a)$. Why?

4. $(P+p) \times \frac{1}{2}(r - a) < P \times (r - a)$. Why?

5. As the number of sides of the polygon is increased, $P$ diminishes, and $(r - a)$ approaches zero as a limit (§ 735). Therefore, $P \times (r - a)$ approaches zero as a

limit. Why? $\therefore A^1 - A$ continually decreases and approaches zero as a limit. Why?

But $M - A$ is always less than $A^1 - A$. As $M - A$ cannot become zero, $M$ is the limit toward which $A$ is approaching.

Apply the four tests of a limit of a variable in drawing the conclusion.

*Therefore —*

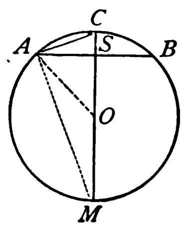
**739.** COROLLARY. *The area of the circle is the limit of the area of a regular circumscribed polygon if the number of sides of the polygon is indefinitely increased.*

---

Ex. 467. Find the locus of a point equally distant from the circumferences of two concentric circles.

Ex. 468. Show how to cut off the corners of an equilateral triangle in such a way that the remaining figure will be a regular hexagon.

Ex. 469. (§ 384.) Another method for § 384.




*To find value of $AC$ in terms of $AB$ and $R$.*

Draw $CM$, a diameter. Connect $A$ and $M$.

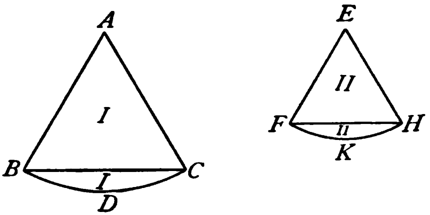
*Suggestion.* Find value of $AC$ in terms of $CS$ and $CM$. § 310.

Ex. 470. Find the area of a circular ring between the circumferences of two concentric circles whose diameters are eight and ten inches, respectively.

Ex. 471. Find the length of a line, Exercise 470, that is the tangent of one circle and the chord of the other.

## Proposition XXIII. (§ 379.)

**740. Theorem.** *The areas of two similar segments have the same ratio as the squares of their radii.*




**Let B D C and F H K, Segment I and Segment II, respectively, represent two similar segments of circles whose centers are A and E.**

*To prove* $\frac{\text{Seg. I.}}{\text{Seg. II.}} = \frac{\overline{AB}^2}{\overline{EF}^2}$.

*Suggestion 1.* Compare $\Delta$ I with $\Delta$ II.

2. $\frac{\text{Sec. I.}}{\text{Sec. II.}} = \frac{\Delta\ \text{I.}}{\Delta\ \text{II.}}$ § 379.

3. Take by alternation and that result by division.

4. $\frac{\text{Seg. I.}}{\text{Seg. II.}} = \frac{\overline{AB}^2}{\overline{EF}^2}$. Why?

*Therefore* —

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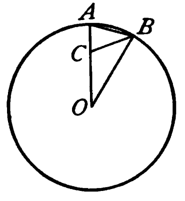
Ex. 472. If three equal circles are tangent to each other what is the area of the surface included between them if the radius of each of the circles is five rods?

Ex. 473. A hexagon whose side is six inches is inscribed in a circle. Find the area of the inscribed triangle.

PROPOSITION XXIV. (§ 390.)

**741. Problem.** *To inscribe a regular decagon in a given circle.*



**Let O be a given circle and O A its radius.**

*To inscribe a regular decagon in circle O.*

*Suggestion 1.* Divide the radius $O A$ in mean and extreme ratio, and let $O C$ be the greater segment.

*2.* Draw the chord $A B$ equal to $O C$ and connect $O$ with $B$, and $C$ with $B$. To prove arc $A B$ = one-tenth of the circumference.

*3.* $\frac{O A}{O C} = \frac{O C}{A C}$. Why? § 723. *4.* $\frac{O A}{A B} = \frac{A B}{A C}$. Why?

*5.* $\triangle O A B$ is similar to $\triangle C A B$. Why? § 303.

*6.* Compare $\angle A B C$ and $\angle O$; $\angle C B O$ with $\angle O$; $\angle A B O$ with $\angle O$.

*7.* $\angle O$ is contained how many times in 2 rt. $\angle$s? In 4 rt. $\angle$s?

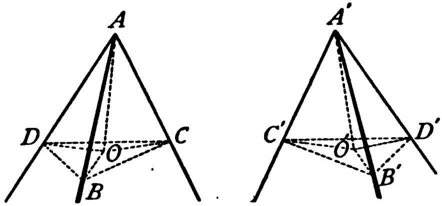
*8.* Hence $A B$ subtends what fractional part of the circumference?

*9.* State method of inscribing a regular decagon in a circle.

COROLLARY. *By joining the alternate vertices of a regular inscribed decagon, a regular inscribed pentagon is formed.*

## PROPOSITION XXV. (§ 477.)

742. **Theorem.** *Two symmetrical trihedral angles are equal in magnitude.*




**Let A and A′ represent two symmetrical trihedral angles.**

*To prove that A and A′ are equal in magnitude.*

*Suggestion 1.* Pass planes cutting off the edges so that $AD$, $AC$, $AB$, $A'B'$, $A'C'$ and $A'D'$ are equal. Drop ⊥s from $A$ and $A'$ to the respective planes, as $AO$ and $A'O'$; connect $O$ with $C$, $D$ and $B$; $O'$ with $C'$, $D'$ and $B'$.

*2.* $A\text{-}DCO$ is an isosceles trihedral angle, symmetrical to $A'\text{-}D'C'O'$. Why? The same is true of $A\text{-}DBO$ and $A'\text{-}D'B'O'$; of $A\text{-}CBO$ and $A'\text{-}C'B'O'$. Give auth.

*3.* Compare $A$ and $A'$.

*Therefore* —

Additional helps: To answer Suggestion 2, compare $< DAO$ and $CAO$, etc.

NOTE.—In dropping the ⊥s $A\ O$ and $A'\ O'$, points $O$ and $O'$ may lie within the perimeters of the △s, upon the perimeters, or without the perimeters. If $DBC$ and $D'B'C'$ are acute triangles, where do $O$ and $O'$ lie? If obtuse triangles, where do they lie? If right triangles, where do they lie? Adapt the demonstration to each situation. Compare this proposition with § 673.

Ex. 474. (§ 558.) To construct a regular tetrahedron.

*Suggestion.* Construct an equilateral △ for the base. At the center of the △ erect a ⊥. With a radius equal to a side of the △ and with a center at one vertex of the △, cut off the perpendicular. Complete the construction.

Ex. 475. To construct a regular hexahedron.

Ex. 476. To construct a regular octahedron.

*Suggestion.* Construct a square, and at its center erect a ⊥. With a vertex of the square as a center, and a radius equal to one side of the square, cut off the ⊥ on each side of the square. Complete the construction.

Ex. 477. To construct a regular icosahedron.

*Suggestion.* Construct a regular pentagon, and at its center erect a ⊥. With a vertex of the pentagon as a center, and a radius equal to one side of the pentagon, cut off the ⊥. Join the point on the ⊥ to each vertex of the pentagon thus forming a pentagonal pyramid. The polyhedral ∠ at the vertex is one of the polyhedral ∠s of the icosahedron required. Construct equal polyhedral ∠s at the vertices of the pentagon first drawn by supplying three additional equilateral triangles, whose sides are each equal to the sides of the pentagon, at each vertex. The exposed edges of the triangles form a pentagon. Close the figure by constructing, as before, a pentagonal pyramid upon this pentagon.

Ex. 478. To construct a regular dodecahedron.

*Suggestions:*

Construct a regular pentagon. At each of its vertices construct a trihedral angle, using equal pentagons for faces. At each vertex of a trihedral angle in the exposed edge of the figure, supply a pentagon for the third face. Close the figure with a pentagon.

**The highest value of these constructions is to build them in the mind without the aid of a physical construction.**

Ex. 479. A regular quadrangular pyramid has each basal edge equal to 12 feet, and each lateral edge equal to 10 feet. Find the altitude of the pyramid.

Ex. 480. Prove that the sum of the angles of a convex spherical polygon of $n$ sides is greater than $2n - 4$ and less than $6n - 12$ right angles.

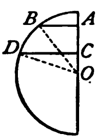
*Suggestion.* Divide the spherical polygon of $n$ sides into spherical triangles by arcs of great circles drawn from one vertex to each non-adjacent vertex. How many $\triangle$s are formed?

Ex. 481. Prove that the number of spherical degrees in a convex spherical polygon of $n$ sides is equal to the excess of the sum of its angles over $2n - 4$ right angles.

See § 690.

## Proposition XXVI. (§ 709.)

**743. Problem.** *To find the volume of a spherical segment.*




**Let A B D C generate a spherical segment by revolving about A C.**

*To find the volume of the segment.*

*Suggestion 1.* Join $B$ and $D$ each to the center of the semi-circle $O$.

*2*. Find the volume of the spherical sector generated by revolving the circular sector *B O D*. § 709.

*3*. Add the volume of the cone generated by revolving the △ *B A O* upon *A O*.

*4*. Subtract the volume of the cone generated by revolving the △ *D C O* upon *C O*. § 612.

**744.** COROLLARY I. ***Find the volume of a spherical segment less than a hemisphere of one base.***

**745.** COROLLARY II. ***Find the volume of a spherical segment of two bases which includes the center of the sphere.***

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Ex. 482. Any straight line drawn through the middle point of any diagonal of a parallelopiped, terminating in two opposite faces, is bisected at that point.

Ex. 483. A right circular cylinder whose altitude is 4 feet and the radius of whose base is 3 feet is cut by a plane parallel to the base at such a distance from the base that the area of the section is a mean proportional between the areas of the convex surfaces of the two parts into which the cylinder is divided. Determine the lengths of the segments of the altitude.

Ex. 484. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron whose faces are respectively similar to the corresponding faces of the given tetrahedron.

Ex. 485. Find the volume of a spherical segment, one base of which passes through the center of the sphere, the radius of the sphere being 20 inches and the altitude of the segment 8 inches.

Ex. 486. A cylinder of revolution, and the frustum of a cone of revolution, have the same lower base and the same altitude. What must be the ratio of the radii of the two bases of the frustum in order that the volume of the frustum may be one-half the volume of the cylinder?

Ex. 487. If a rectangle revolves about one of its sides, the volume generated is 288 $\pi$ cubic feet, but if it revolves about the adjacent side, the volume generated is 384 $\pi$ cubic feet. Find the diagonal of the rectangle.

Ex. 488. Find the radius of the base of a cylinder of revolution whose altitude is 1 yard and whose volume is 48 $\pi$ cubic feet.

Ex. 489. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

Ex. 490. The sum of the squares of the twelve edges of a parallelopiped equals the sum of the squares upon its four diagonals.

Ex. 491. If the diameter of the sphere is 30 feet, find the volumes of the first, fifth, and seventh segments referred to in exercise 413.

# THE INDEX.

*References are to Pages.*